

Ordinal optimization - empirical large deviations rate estimators, and stochastic multi-armed bandits

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July 17, 2015

Abstract

Consider the ordinal optimization problem of finding a population amongst many with the smallest mean when these means are unknown but population samples can be generated via simulation. Typically, by selecting a population with the smallest sample mean, it can be shown that the false selection probability decays at an exponential rate. Lately researchers have sought algorithms that guarantee that this probability is restricted to a small δ in order $\log(1/\delta)$ computational time by estimating the associated large deviations rate function via simulation. We show that such guarantees are misleading. Enroute, we identify the large deviations principle followed by the empirically estimated large deviations rate function that may also be of independent interest. Further, we show a negative result that when populations have unbounded support, any policy that asymptotically identifies the correct population with probability at least $1 - \delta$ for each problem instance requires more than $O(\log(1/\delta))$ samples in making such a determination in any problem instance. This suggests that some restrictions are essential on populations to devise $O(\log(1/\delta))$ algorithms with $1 - \delta$ correctness guarantees. We note that under restriction on population moments, such methods are easily designed. We also observe that sequential methods from stochastic multi-armed bandit literature can be adapted to devise such algorithms.

1 Introduction

Suppose that we can sample independently from d different random variables or ‘populations’ $(X(i) : i \leq d)$. Further, from each population i , we can generate independent identically distributed (iid) samples $(X(i, j) : j \geq 1)$. The distribution of $(X(i) : i \leq d)$ is not known and our aim is to find the ‘best’ population

$$i^* = \arg \min_{1 \leq j \leq d} EX(j).$$

Ordinal optimization corresponds to observing that under light-tailed assumption on the distribution of $(X(i) : i \leq d)$, ordinals are learnt correctly faster than the values of expectations. (We say that a distribution is light-tailed if its moment generating function is finite in a neighbourhood of zero). More specifically,

$$P(\bar{X}_n(i^*) > \min_{j \neq i^*} \bar{X}_n(j))$$

decays exponentially in n , where $\bar{X}_n(i) = \frac{1}{n} \sum_{k=1}^n X(i, k)$, while it is well known through central limit theorem that typically rate of convergence of $\bar{X}_n(i)$ to $EX(i)$ is $n^{-1/2}$. This suggests that for small but positive δ , one may be able to construct algorithms that generate $O(\log(1/\delta))$ samples of $X(i)$'s and make a correct selection while restricting the probability of false selection to δ . In this paper we critically examine this proposition, answering this in negative in a general light-tailed settings, and in positive when further information on moments of the underlying random variables is available. Applications of ordinal optimisation in simulation arise in selecting a best design from a set of competing designs via simulation where all of the designs may be modelled as discrete event dynamic systems. Such systems include queueing systems, computer and communications networks, manufacturing systems and transportation networks (see, e.g., Ho, Srinivas and Vakili 1992 for some applications).

1.1 Brief literature review

A brief literature survey is in order: Ho et. al. (1992) observed that determining ordinals amongst population means is faster than estimating the means. Dai (1996) used large deviations to show in a fairly general framework that for light-tailed random variables the probability of false selection decays exponentially. Chen, Lin, Yucsan and Chick (2000) considered the problem of ordinal optimization under the assumption that a fixed but large computation budget n is available and the underlying random variables have a Gaussian distribution. They attempted to optimize the budget allocated to each population asymptotically as $n \rightarrow \infty$ so that the probability of false selection is minimized. Glynn and Juneja (2004) observed that for $p_i > 0$,

$$P(\bar{X}_{i^*}(p_{i^*}n) > \min_{i \neq i^*} \bar{X}_i(p_i n)) \leq e^{-nH(p_1, \dots, p_d)}, \quad (1)$$

and $\lim_{n \rightarrow \infty} \frac{1}{n} P(\bar{X}_{i^*}(p_{i^*}n) > \min_{i \neq i^*} \bar{X}_i(p_i n)) = -H(p_1, \dots, p_d)$. They, then optimised this large deviations rate function $H(p_1, \dots, p_d)$ under the constraint $\sum_{i=1}^d p_i = 1$ to determine the optimal allocations as $n \rightarrow \infty$ even for non-Gaussian distributions. Significant literature since then has appeared that relies on large deviations analysis (e.g., Hunter and Pasupathy 2013, Szechtman and Yucsan 2008, Broadie, Han, Zeevi 2007, Blanchet, Liu, Zwart 2008).

Substantial literature exists on selecting the best system amongst many alternatives using ranking/selection procedures (see, e.g., Kim and Nelson 2001, 2003, Nelson et. al. 2001, Branke, Chick and Schmidt 2007 for an overview). Gaussian assumption is critical to most of the analysis here. These approaches also consider the ‘indifference-zone formulation’ where it is assumed that there exists a known $\epsilon > 0$ such that

$$EX(i^*) \leq EX(j) - \epsilon \quad (2)$$

for $j \neq i^*$ (see, e.g., Nelson and Matejcek 1995). Such an ϵ is then useful in devising rules for the number of samples needed to control the probability of false selection to pre-specified levels.

Ordinal optimisation methods are also related to the vast, elegant and evolving literature in learning and statistics community referred to as stochastic multi-armed bandit methods. Typically, this strand of literature refers to sampling from a population as ‘pulling an arm’ and assumes that each such pull leads to a random reward whose distribution depends upon the population. The aim then is to develop optimal or near optimal sequential sampling strategy that maximises the long term total expected reward or equivalently minimises the total expected regret (regret over n trials is referred to as the reward that would have been realised in these trials if the ‘best’ arm was pulled each time versus the actual reward realisation from a given sequential strategy - here best refers to the arm with the highest expected reward. See, e.g., Bubeck and Cesa-Bianchi 2012, Cesa-Bianchi and Lugosi 2006 for surveys on these methods; Lai and Robbins 1985 for a seminal paper in this area). Optimal sequential strategies have to carefully manage the exploration-exploitation trade-offs in arm selections. Recently, these methods have also been used in the *pure exploration setting* where the goal is to identify the arm with the highest expected reward in minimum expected number of trials. See, e.g., Even-Dar, Mannor and Mansour (2002, 2006), Audibert and Bubeck (2010), Jamieson, K., Malloy, M., Nowak, R., & Bubeck, S. (2013). Thus, this problem is identical to the ordinal optimisation problem that we consider. A standard assumption both in minimising regret and the pure exploration settings is that the rewards from each arm are either Bernoulli or are bounded with known bounds. In a recent paper Bubeck, Cesa-Bianchi and Lugosi (2013), again consider the problem of minimising expected regret under the assumption that the rewards are unbounded but explicit bounds on their moments are known.

1.2 Observations

Observe that if, as in (1), $P(FS) \leq e^{-nI}$, for some $I > 0$, then

$$n = \frac{1}{I} \log(1/\delta) \text{ ensures } P(FS) \leq \delta.$$

However this relies on estimating I from the samples generated. One hopes for algorithms that for $n = O(\log(1/\delta))$ ensure that at least asymptotically $P(FS) \leq \delta$, that is,

$$\limsup_{\delta \rightarrow 0} P(FS)\delta^{-1} \leq 1 \quad (3)$$

even when an indifference zone formulation is considered so that there exists a fixed and known $\epsilon > 0$ and (2) holds.

Note that $O(\log(1/\delta))$ effort is necessary to achieve (3) in the sense that if $\log(1/\delta)^{1-\tilde{\epsilon}}$ samples are generated, for any small $\tilde{\epsilon} > 0$, then

$$P(X_i \in A)^{\log(1/\delta)^{1-\tilde{\epsilon}}} = \delta^{\frac{\text{positive no.}}{\log(1/\delta)^{\tilde{\epsilon}}}} > \delta$$

as $\delta \rightarrow 0$.

Also observe that, $O(\log(1/\delta)^{1+\tilde{\epsilon}})$ is sufficient as

$$\delta^{-1}P(FS) \leq \delta^{-1}e^{-nI} = \delta^{-1}e^{-\log(1/\delta)^{1+\tilde{\epsilon}}I} = \delta^{\log(1/\delta)^{\tilde{\epsilon}}I-1}$$

which goes to zero as $\delta \rightarrow 0$.

1.3 Our contributions

We first consider two practically reasonable implementations that involve estimating the large deviations rate function I associated with the probability of false selection from $O(\log(1/\delta))$ generated samples and using this estimator as a proxy for I to control $P(FS)$. We argue that there exist light tailed distributions for which such methods would fail. Enroute, we conduct large deviations analysis of the empirically estimated large deviations rate function. This is useful to our analysis and may also be of independent interest.

Our key result is negative and is as follows - Given any (ϵ, δ) algorithm that correctly separates populations with mean difference at least ϵ with

$$\limsup_{\delta \rightarrow 0} P(FS)\delta^{-1} \leq 1,$$

we prove that for populations with unbounded support, under mild restrictions, the expected number of samples cannot be $O(\log(1/\delta))$. This result also holds for restrictive (ϵ, δ) policies where $P(FS) \leq \delta$ for any given δ , not just asymptotically. Further, similar results also hold when the criteria for selecting the best design may not be the population mean but another function of the population distribution such as its specific quantile.

Our positive contributions - Under explicitly available upper bounds on convex, increasing functions of underlying random variables, we develop random variable truncation

as well as capping based $O(\log(1/\delta))$ computation time (ϵ, δ) algorithms. We also observe that the recently proposed sequential algorithms in multi-armed bandit regret setting (see Bubeck et. al. 2013) when heavy tails are involved, are easily adapted to this *pure exploration setting* to provide $O(\log(1/\delta))$ computation time sequential algorithms. We also develop upper bounds on computational effort under these algorithms and suggest tweaks that may lead to minor performance improvements.

1.4 Roadmap

In Section 2 we review some basic large deviations results and conduct large deviations analysis of the empirical large deviations estimator. The results are useful to our analysis in Sections 3 and 4, where we discuss pitfalls of some standard approaches for selecting the best system that rely on empirical estimator of the large deviations rate function. Specifically, in Section 3, we consider a standard two phase approach to ordinal optimisation adapted to rely on estimating the large deviations rate function, and point out its drawbacks. In Section 4, we consider a reasonable sequential approach that relies on estimated large deviations rate function and illustrate cases where it may fail. Section 5 contains our key negative result illustrating that, under mild regularity conditions, the impossibility of algorithms that run in $O(\log(1/\delta))$ time and control the probability of false selection to within small δ . In Section 6 and 7 we provide some positive results. In Section 6, we develop $O(\log(1/\delta))$ algorithms when upper bounds on suitable moments of underlying random variables are available. In Section 7, we adapt recent results from multi-armed bandit related research to our ordinal optimisation setting.

All the proofs are given in the appendix.

Note that this is a rough draft prepared to facilitate early dissemination. Versions with improved structure and fewer errors should (hopefully!) update this draft soon.

2 Large deviations analysis of empirical large deviations estimator

We first review the basic Cramer's theorem that is critical to our analysis. Suppose X_1, X_2, \dots, X_n are i.i.d. samples of X and $a > EX$. Let $\Lambda(\theta) = \log Ee^{\theta X}$ denote the log moment generating function of X and let

$$I(a) = \sup_{\theta \in \mathfrak{R}} (\theta a - \Lambda(\theta)),$$

denote its Legendre transform. Then it follows from Cramer's theorem that

$$P(\bar{X}_n \geq a) \leq \exp(-n \inf_{x \geq a} I(x)) = \exp(-nI(a)).$$

Furthermore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log P(\bar{X}_n \geq a) = -I(a).$$

$I(\cdot)$ then denotes the the large deviations rate function of X . It is a nonnegative convex function that equals 0 at EX .

As mentioned earlier, in Sections 3 and 4 we point out some drawbacks of using some standard approaches to control the probability of false selection that rely on estimating the large deviations rate function. To keep the discussion simple there and in this section, we consider the case where $d = 2$ and equal number of samples are allocated to each population. This case further simplifies to that of a single random variable X with an unknown mean EX and our interest is in establishing either $EX > 0$ or $EX < 0$ with error probability $\leq \delta$, asymptotically as $\delta \rightarrow 0$.

Note that this determination may be based on the sign of the sample average \bar{X}_n of n iid samples of X . Then, if $EX < 0$, we may take $\exp(-nI(0))$ as a proxy for $P(\bar{X}_n \geq 0)$, the probability of false selection. Similarly, if $EX > 0$, we may again take $\exp(-nI(0))$ as a proxy for $P(\bar{X}_n \leq 0)$, the probability of false selection. Thus, $\lceil \frac{\log(1/\delta)}{I(0)} \rceil$ samples ensure that $P(FS) \leq \delta$. Since $I(0)$ is typically unknown, an estimator needs to be ascertained from the empirically generated samples.

2.1 Empirical estimator for $I(0)$

Without loss of generality we assume that $EX < 0$ and that random variable X is not degenerate. Recall that $\Lambda(\theta)$ denotes the log-moment generating function $\log E \exp(\theta X)$. It is a strictly convex with $\Lambda(0) = 0$ and $\Lambda'(0) = EX < 0$. Furthermore,

$$I(0) = -\inf_{\theta} \Lambda(\theta).$$

A natural estimator for $I(0)$ based on samples $(X_i : 1 \leq i \leq m)$ is

$$\hat{I}_m(0) = -\inf_{\theta \in \mathcal{R}} \hat{\Lambda}_m(\theta)$$

where

$$\hat{\Lambda}_m(\theta) = \log \left(\frac{1}{m} \sum_{i=1}^m \exp(\theta X_i) \right)$$

We now conduct large deviations analysis of the estimator $\hat{I}_m(0)$. Duffy and Metcalfe (2005) conduct large deviations analysis for the empirical rate function on the functional space. Our analysis focusses on the empirical rate function evaluated at a specified value and provides greater intuitive insight into the large deviations event.

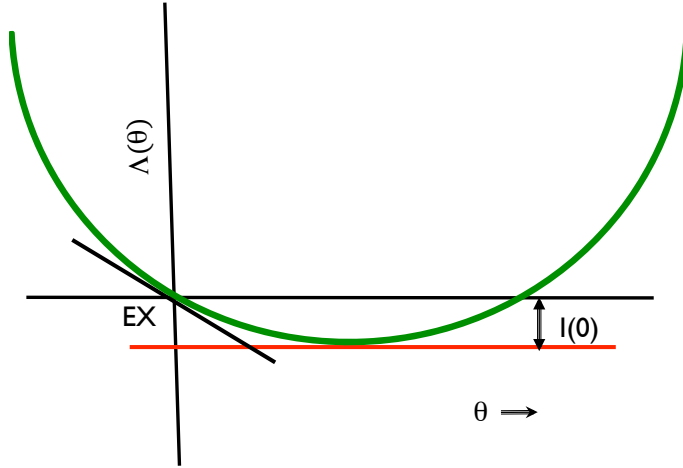


Figure 1: Graphical description of the log-moment generating function $\Lambda(\cdot)$ and the large deviations rate function at zero $I(0) = -\inf_{\theta} \Lambda(\theta)$.

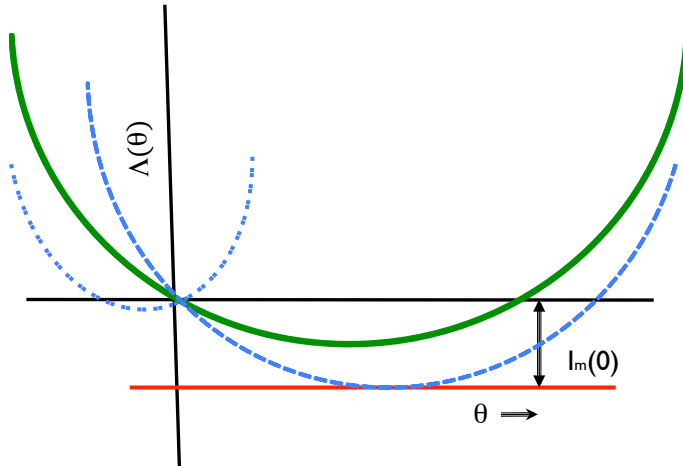


Figure 2: Examples of empirically estimated log-moment generating functions. One where the sample mean is negative, other where it is positive.

Reader more interested in the ordinal optimisation discussion may skip the discussion below and go directly to Section 3 at initial reading. Readers willing to ignore the large deviations analysis may go directly to Section 5 for our key negative result and to Sections 6 and 7 for more positive conclusions.

2.2 Large deviations analysis for $\hat{I}_m(0)$

Theorem 1 below states a large deviations result for the empirically estimated large deviations rate function estimator. We need the following light-tailed assumption on the distribution of X .

Assumption 1 *For any $K > 0$, there exists a $u > 0$ such that*

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\bar{X}_m > u) \leq -K$$

and

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\bar{X}_m < -u) \leq -K.$$

Theorem 1 *Under Assumption 1, for $a > I(0)$,*

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{I}_m(0) \geq a) = \lim_{m \rightarrow \infty} \frac{1}{m} \log P\left(\inf_{\theta \in \mathbb{R}} \frac{1}{m} \sum_{i=1}^m e^{\theta X_i} \leq e^{-a}\right) = -\inf_{\theta \in \mathbb{R}} \mathcal{I}_\theta(e^{-a}), \quad (4)$$

where

$$\mathcal{I}_\theta(\nu) = \sup_{\alpha \in \mathbb{R}} (\alpha \nu - \log E \exp(\alpha e^{\theta X})).$$

Corollary 1 *From the proof of Theorem 1 it is easily seen that for $a > -\inf_{\theta \in A} \Lambda(\theta)$,*

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log P\left(\inf_{\theta \in A} \hat{\Lambda}_m(\theta) \leq -a\right) = -\inf_{\theta \in A} \mathcal{I}_\theta(e^{-a}) \quad (5)$$

for any closed interval A .

2.2.1 $P(\hat{I}_m(0) \leq a)$ for $0 < a < I(0)$

The large deviations upper bound for $P(\hat{I}_m(0) \leq a)$, for $0 < a < I(0)$, is easily seen. Let $\Theta_a = \{\theta : \Lambda(\theta) \leq -a\}$. It is clearly a non-empty interval. Note that

$$P(\hat{I}_m(0) \leq a) = P(\inf_{\theta \in \mathbb{R}} \hat{\Lambda}_m(\theta) \geq -a) \leq \inf_{\theta \in \mathbb{R}} P(\hat{\Lambda}_m(\theta) \geq -a) \leq \inf_{\theta \in \mathbb{R}} \exp(-m \inf_{x \geq e^{-a}} \mathcal{I}_\theta(x)).$$

Note that $\mathcal{I}_\theta(e^{\Lambda(\theta)}) = 0$. So

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{I}_m(0) \leq a) \leq -\sup_{\theta \in \Theta_a} \mathcal{I}_\theta(e^{-a})$$

follows.

The lower bound requires greater technicalities. To keep the analysis simple we make Assumption 2 below. Let $\mathcal{D}_\Lambda = \{\theta : \Lambda(\theta) < \infty\}$.

Assumption 2 For $0 < a < I(0)$, let $\Theta_a = [\underline{\theta}_a, \bar{\theta}_a]$ and $\Theta_a \subset \mathcal{D}_\Lambda^o$.

Assumption 2 implies that $\underline{\theta}_a > 0$ and $\Lambda(\underline{\theta}_a) = \Lambda(\bar{\theta}_a) = -a$.

Below in Proposition 1 we consider the case where $P(X \geq x)$ for large x is bounded from below by a term exponentially decaying in x . Theorem 2 later considers the case where this is not true.

Proposition 1 Suppose that Assumption 2 holds, and

$$P(X \geq x) \geq \exp(-\lambda x)$$

for some $\lambda > 0$ and all x sufficiently large. Then,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{I}_m(0) \leq a) = 0. \quad (6)$$

Furthermore,

$$\sup_{\theta \in \Theta_a} \mathcal{I}_\theta(e^{-a}) = 0. \quad (7)$$

We now consider $P(\hat{I}_m(0) \leq a)$, $0 < a < I(0)$, $\mathcal{I}_\theta(e^{-a}) > 0$ for $\theta \in (\underline{\theta}_a, \bar{\theta}_a)$. Specifically, we require Assumption 3 below for further analysis. Let

$$\mathcal{H} = \{(\alpha, \theta) : E \exp(\alpha e^{\theta X}) < \infty\}.$$

Assumption 3 For $0 < a < I(0)$, there exist positive $(\alpha^*, \theta^*) \in \mathcal{H}^o$ that uniquely maximize

$$f(\alpha, \theta) \triangleq \alpha e^{-a} - \log E \exp(\alpha e^{\theta X}). \quad (8)$$

Remark 1 It is easy to check that (α^*, θ^*) satisfy the following first order conditions for maximising $f(\alpha, \theta)$,

$$e^{-a} = \frac{E e^{\theta^* X} \exp(\alpha^* e^{\theta^* X})}{E \exp(\alpha^* e^{\theta^* X})}, \quad (9)$$

and

$$E X e^{\theta^* X} \exp(\alpha^* e^{\theta^* X}) = 0. \quad (10)$$

Also note that for $\theta \in \Theta_a^o$,

$$\mathcal{I}_\theta(e^{-a}) = \alpha(\theta) e^{-a} - \log E[\exp(\alpha(\theta) e^{\theta X})],$$

where $\alpha(\theta)$ uniquely solves

$$e^{-a} = \frac{E e^{\theta X} \exp(\alpha e^{\theta X})}{E \exp(\alpha e^{\theta X})}.$$

Furthermore, differentiating it and setting the derivative equal to zero, it can be seen that

$$\sup_{\theta \in \Theta_a} \mathcal{I}_\theta(e^{-a}) = \mathcal{I}_{\theta^*}(e^{-a}) = \alpha^* e^{-a} - \log E[\exp(\alpha^* e^{\theta^* X})].$$

Theorem 2 *Under Assumptions 2 and 3, for $0 < a < I(0)$,*

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{I}_m(0) \leq a) = -\mathcal{I}_{\theta^*}(e^{-a}) = -\sup_{\theta \in \Theta_a} \mathcal{I}_\theta(e^{-a}). \quad (11)$$

Remark 2 Large deviations analysis for empirical large deviations rate estimator of $I(x)$ follows analogously. Note that $I(x)$ is just the rate function for $X - x$ evaluated at zero. A natural estimator for $I(x)$ is

$$\hat{I}_m(x) = \sup_{\theta \in \mathbb{R}} (\theta x - \hat{\Lambda}_m(\theta)).$$

Furthermore, under similar technical conditions, for $a > I(x)$,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{I}_m(x) \geq a) = \lim_{m \rightarrow \infty} \frac{1}{m} \log P\left(\inf_{\theta \in \mathbb{R}} \frac{1}{m} \sum_{i=1}^m e^{\theta(X_i - x)} \leq e^{-a}\right) = -\inf_{\theta \in \mathbb{R}} \mathcal{I}_\theta(e^{-a+\theta x}). \quad (12)$$

Further, for $a < I(x)$

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{I}_m(x) \leq a) = -\sup_{\theta \in [\underline{\theta}_a, \bar{\theta}_a]} \mathcal{I}_\theta(e^{-a+\theta x}), \quad (13)$$

where now $0 < \underline{\theta}_a < \bar{\theta}_a$ are such that $\Lambda(\underline{\theta}_a) = -a + \underline{\theta}_a x$ and $\Lambda(\bar{\theta}_a) = -a + \bar{\theta}_a x$.

3 Two phase ordinal optimisation implementation

In this section we consider a two phase procedure to determine whether $EX > 0$ or < 0 . In the first phase, using $O(\log(1/\delta))$ samples, $I(0)$ is estimated. In the second phase, this estimator is used as a proxy for $I(0)$ in deciding the number of samples to generate. We also illustrate the poor performance of this approach in some settings.

The specific two phase procedure is as follows:

- **First phase** - Generate $m = \lceil c_1 \log(1/\delta) \rceil$ independent samples of X to estimate $I(0)$ by $\hat{I}_m(0)$ for some $c_1 > 0$.

- **Second phase** - Generate

$$\lceil c_2 c_1 \log(1/\delta) / \hat{I}_m(0) \rceil = \lceil c_2 m / \hat{I}_m(0) \rceil \triangleq N$$

independent samples of X for some $c_2 > 0$.

- Decide the sign of EX based on whether $\bar{X}_N > 0$ or $\bar{X}_N \leq 0$.

Remark 3 In the above procedure,

$$P(FS) = P\left(\sum_{i=1}^N X_i > 0\right) \approx E \exp\left(-\frac{c_2 m}{\hat{I}_m(0)} I(0)\right).$$

Thus, large values of $\hat{I}_m(0)$ lead to undersampling in second phase and contribute the most to $P(FS)$. Note that individual X_i 's taking large values do not contribute significantly to the estimator

$$\hat{I}_m(0) = -\inf_{\theta \in \mathfrak{R}} \log\left(\frac{1}{m} \sum_{i=1}^m \exp(\theta X_i)\right)$$

taking large values. This may appear counter-intuitive when $P(X > x) \sim \exp(-\lambda x)$ for large x , $\lambda > 0$, as then $\exp(\theta X_i)$ are heavy-tailed (see, e.g., Foss, Korshunov and Zachary 2011 for an introduction to heavy-tailed distributions) and it is well known that large deviations of $\frac{1}{m} \sum_{i=1}^m \exp(\theta X_i)$, i.e., $\frac{1}{m} \sum_{i=1}^m \exp(\theta X_i)$ taking an unusually large value, is governed by the largest term in the sum. However, $\hat{I}_m(0)$ takes unusually large values when $\inf_{\theta \in \mathfrak{R}} \frac{1}{m} \sum_{i=1}^m \exp(\theta X_i)$ takes unusually small values, and here large values taken by individual X_i 's have little impact.

Theorem 3 For $N = \lceil m / \hat{I}_m(0) \rceil$, under Assumption 1,

$$\liminf_m \frac{1}{m} \log P\left(\sum_{i=1}^N X_i > 0\right) \geq -\inf_{b \geq 0} \left(\frac{c_2 I(0)}{b} + \inf_{\theta} \mathcal{I}_{\theta}(e^{-b})\right). \quad (14)$$

In particular, for $c_1 = c_2 = 1$,

$$\liminf_{\delta \rightarrow 0} P\left(\sum_{i=1}^N X_i > 0\right) \delta^{-1} > 1. \quad (15)$$

Remark 4 Consider the term

$$c_1 \inf_{\gamma \geq 0} \left(c_1 \frac{I(0)}{\gamma} + \inf_{\theta} \mathcal{I}_{\theta}(e^{-\gamma})\right). \quad (16)$$

It is easy to see that since

$$\mathcal{I}_{\theta}(e^{-\gamma}) = \sup_{\alpha \in \mathfrak{R}} \left(-\alpha e^{-\gamma} - \log E \exp(-\alpha e^{\theta X})\right),$$

then the solutions θ and α to $\inf_{\theta} \mathcal{I}_{\theta}(e^{-\gamma})$ uniquely solve the equations

$$e^{-\gamma} = \frac{Ee^{\theta X} \exp(-\alpha e^{\theta X})}{E \exp(-\alpha e^{\theta X})} \quad (17)$$

and

$$EXe^{\theta X} \exp(-\alpha e^{\theta X}) = 0. \quad (18)$$

Combining this in (16), it can be seen that (γ, θ, α) that optimise (16), uniquely solve (17), (18) and

$$\alpha e^{-\gamma} = I(0)/\gamma^2. \quad (19)$$

Now consider a rv X that takes values $-b$ and b , $b > 0$, with probability p and $1 - p$, respectively with $p > 0.5$. Clearly then $EX = -(2p - 1)b < 0$. Solving equations (17, 18, 19) it can be seen that RHS in (14) is independent of value b - hence any indifference zone considerations can be met by scaling b no matter how close p is to 0.5.

For $c_1, c_2 = 1$, and $p = 0.55, 0.52, 0.51$, RHS in (14) equals 0.105, 0.047 and 0.025, respectively. It can be seen that as $p \downarrow 0.5$, k, θ and RHS in (14) decrease to zero, and $\alpha \rightarrow 1$.

4 Sequential ordinal optimisation implementation

We now critique a natural sequential procedure for deciding whether $EX > 0$ or < 0 that relies on large deviations rate function estimator $\hat{I}_m(0)$ for a stopping rule and that uses $O(\log(1/\delta))$ computational effort and attempts to control the probability of false selection to within δ for small δ . Further, we identify some light-tailed distributions for which the algorithm behaves poorly. (See, e.g., Branke, Chick and Schimdt 2007, Goldsman et al. 2002 for sequential procedures used under Gaussian assumption in ranking and selection simulation literature).

Again, without loss of generality we assume that $EX < 0$. Recall that in this case $\exp(-m\hat{I}_m(0))$ is a reasonable proxy for the probability of false selection.

Consider the following procedure:

- For $c_1 > 0$, generate $m_1 = c_1 \log(1/\delta)$ independent samples $(X_i : i \leq m_1)$ of X in the first phase to estimate $I(0)$ by $\hat{I}_{m_1}(0)$.
- If $\exp(-m_1\hat{I}_{m_1}(0)) \leq \delta$, terminate, and conclude that the sign of EX is given by the sign of \bar{X}_{m_1} .
- Else, generate another $c_2 \log(1/\delta)$ independent samples of X , where c_2 may be determined adaptively based on the outcome of first m_1 samples. Set $m_2 = (c_1 + c_2) \log(1/\delta)$ and again terminate if $\exp(-m_2\hat{I}_{m_2}(0)) \leq \delta$.

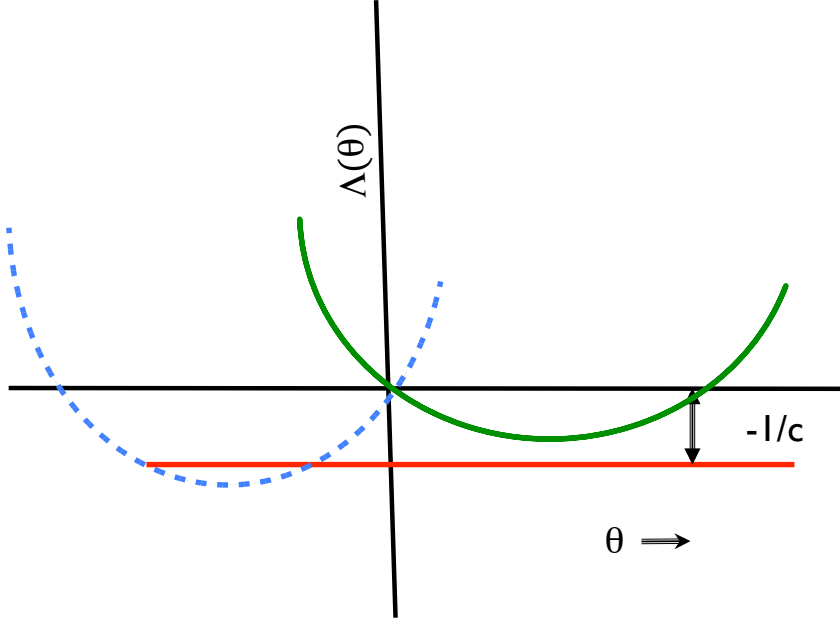


Figure 3: This figure illustrates an empirical log-moment-generating function that would lead to a wrong conclusion (i.e., $EX > 0$) in the sequential algorithm.

- This procedure may then be repeated with c_3, c_4 and so on until the termination criteria is met.

We now identify distributions for which this would wrongly terminate in the first phase itself even for large values of c_1 . Recall that FS denotes the event that the proposed algorithm concludes falsely.

In such a case, we clearly have $I(0) < 1/c_1$. Else,

$$P(\bar{X}_{m_1} \geq 0) \leq \exp(-m_1 I(0)) \leq \delta.$$

In particular, we have $0 > \inf_{\theta} \Lambda(\theta) > -1/c_1$.

Proposition 2 *Suppose that there exists a $\theta < 0$ such that*

$$\mathcal{I}_{\theta}(e^{-1/c_1}) < 1/c_1. \tag{20}$$

Then,

$$\liminf_{\delta \rightarrow \infty} \delta^{-1} P(FS) = \infty. \tag{21}$$

Remark 5 To see numerical examples where (20) holds, consider a random variable $X = K - Y$ where K is a constant, Y is exponentially distributed with rate λ such that $\lambda K < 1$ so that $EX < 0$.

Note that for $\theta > 0$, $\nu < e^{\Lambda(-\theta)}$, hence,

$$\mathcal{I}_{-\theta}(\nu) = \sup_{\alpha \geq 0} (-\alpha\nu - \log E[\exp(-\alpha \exp(-\theta K) \exp(\theta Y))]).$$

This in turn can be seen to equal

$$\sup_{\alpha \geq 0} (-\alpha \exp(\theta K) \nu - \log E[\exp(-\alpha \exp(\theta Y))]).$$

Let $Z(\theta) = \exp(\theta Y)$. Then, it is easily seen that the density function of $Z(\theta)$ at $z \geq 1$ equals

$$\frac{\lambda}{\theta} z^{-(\lambda/\theta+1)}$$

Using this, we get

$$\mathcal{I}_{-\theta}(e^{-1/c_1}) = -\alpha^* \exp(\theta K - 1/c_1) - \log \frac{\lambda}{\theta} \int_1^\infty \exp(-\alpha^* z) z^{-(\lambda/\theta+1)} dz,$$

where α^* solves the equation

$$\exp(\theta K - 1/c_1) = \frac{\int_1^\infty \exp(-\alpha z) z^{-\lambda/\theta} dz}{\int_1^\infty \exp(-\alpha z) z^{-(\lambda/\theta+1)} dz}. \quad (22)$$

Now fixing $\lambda = 1$, $K = 0.96$, we numerically see that for $c_1 = 2$, $\theta = 2.133$, $\alpha^* = 0.0607$ solves (22) with $\mathcal{I}_{-\theta}(e^{-1/c_1}) = 0.2231 < 1/c_1$. For $c_1 = 5$, $\theta = 0.987$, $\alpha^* = 0.201$ solves (22) with $\mathcal{I}_{-\theta}(e^{-1/c_1}) = 0.1259 < 1/c_1$. For $c_1 = 100$, $\theta = 0.129$, $\alpha^* = 1.1792$, solves (22) with $\mathcal{I}_{-\theta}(e^{-1/c_1}) = 0.005425$.

5 The key negative result

In this section we consider for simplicity a two population situation (the case $d > 2$ populations is easily handled) and assume that random outputs from both populations belong to \mathcal{L} , where \mathcal{L} denotes any collection of distribution functions G with finite mean.

Consider a policy $\mathcal{P}(\epsilon, \delta)$ operating on distributions in \mathcal{L} . By this we mean that given $F, G \in \mathcal{L}$ such that the absolute value of the difference of their mean values exceeds $\epsilon > 0$, the policy $\mathcal{P}(\epsilon, \delta)$, generates samples from the two distributions, adaptively (based on the values of generated samples) deciding which distribution to sample from next and at some stage (at a stopping time) selects one of the two populations as the one with the lower mean. The policy guarantees that the probability of false selection $P(FS) \leq \delta$. In this section

we consider a bigger set of policies $\mathcal{P}_{asympt}(\epsilon, \delta)$ that differ from $\mathcal{P}(\epsilon, \delta)$ in that they only guarantee that $P(FS)$ is asymptotically bounded from above by δ . That is,

$$\limsup_{\delta \rightarrow 0} P(FS)\delta^{-1} \leq 1.$$

In many applications, it may be difficult to design a $\mathcal{P}(\epsilon, \delta)$ policy, however, designing a $\mathcal{P}_{asympt}(\epsilon, \delta)$ policy may be easier, and often a practitioner may be satisfied with asymptotic guarantees. (This is analogous to accepting central limit theorem based confidence intervals for population means where the coverage guarantees are valid only asymptotically, see, e.g., Glynn and Whitt 1992). Note that $\mathcal{P}(\epsilon, \delta)$ policies are a subset of $\mathcal{P}_{asympt}(\epsilon, \delta)$.

5.1 Analysis

Recall that the Kullback-Leibler distance between distributions G and \tilde{G} is defined as

$$I(G, \tilde{G}) = \int_{x \in \mathfrak{R}} \left(\log \frac{dG}{d\tilde{G}}(x) \right) dG(x).$$

Suppose that sequences of random variables $(X_i : i \geq 1)$ and $(Y_i : i \geq 1)$ defined on a probability space are i.i.d. and independent of each other under probability measures P_a and P_b .

Under probability measure P_a ,

- distribution of X_i is F with mean μ_F ,
- that of Y_i is G with mean μ_G , where $\mu_G < \mu_F - \epsilon$ ($\epsilon > 0$).

Under P_b ,

- the distribution of X_i is F ,
- that of Y_i is \tilde{G} , where under \tilde{G} , $\mu_{\tilde{G}} > \mu_F + \epsilon$, and G and \tilde{G} are absolutely continuous w.r.t. each other (denoted by $G \sim \tilde{G}$).
- Further, the Kullback-Leibler distance $I(G, \tilde{G}) < \infty$.

The policy $\mathcal{P}_{asympt}(\epsilon, \delta)$ operating on the populations $(X_i : i \geq 1)$ and $(Y_i : i \geq 1)$, for every $\delta > 0$, adaptively generates samples $(X_i : i \leq T_1(\delta))$ and $(Y_i : i \leq T_2(\delta))$ before concluding rv $J(\delta) = 1$, if $\{X_i\}$ is deemed to have lower mean, else $J(\delta) = 2$. Let E_a (E_b) denote the expectation operator under P_a (P_b).

Under P_a , policy $\mathcal{P}_{asympt}(\epsilon, \delta)$ will erroneously conclude that distribution F has the lower mean with error probability $P_a(J(\delta) = 1)$ such that

$$\limsup_{\delta \rightarrow 0} P_a(J(\delta) = 1)\delta^{-1} \leq 1.$$

Under P_b it will erroneously conclude that \tilde{G} has the lower mean with error probability satisfying the relation

$$\limsup_{\delta \rightarrow 0} P_b(J(\delta) = 2)\delta^{-1} \leq 1.$$

The proof of Lemma 1 is similar in spirit to analogous results in Lai and Robbins (1985) and Tsitsiklis and Mannor (2002).

Lemma 1 *Under $\mathcal{P}_{asympt}(\epsilon, \delta)$ operating on \mathcal{L} ,*

$$\liminf_{\delta \rightarrow 0} \frac{E_a T_2(\delta)}{\log(1/\delta)} \geq \sup_{\tilde{G} \in \mathcal{L}, \mu_{\tilde{G}} > \mu_F + \epsilon, \tilde{G} \sim G} \frac{1}{3I(G, \tilde{G})}. \quad (23)$$

Lemma 2 illustrates that if G has unbounded support on the positive real line, that is, $G(x) < 1$ for all $x \in \mathfrak{R}$, then one can always find a \tilde{G} with mean arbitrarily high so that $I(G, \tilde{G})$ is arbitrarily small.

Lemma 2 *Given G with finite mean μ_G and unbounded support on the positive real line, for any $\alpha > 0$, and $k > \mu_G$ there exists a distribution $G_k \in \mathcal{L}$, such that the Kullback-Leibler distance between G and G_k ,*

$$I(G, G_k) = \int_{x \in \mathfrak{R}} \log \left(\frac{dG(x)}{dG_k(x)} \right) dG(x) \leq \alpha \quad (24)$$

and

$$\mu_{G_k} = \int_{x \in \mathfrak{R}} x dG_k(x) \geq k. \quad (25)$$

Theorem 4 below is the main result of this section. Let $\tilde{\mathcal{L}}$ denote a collection of probability distributions on the real line with finite mean, that are not bounded from above and that for any $\alpha, k > 0$, and a $G \in \tilde{\mathcal{L}}$, include a $G_k \in \tilde{\mathcal{L}}$ satisfying (24) and (25). One example of such an $\tilde{\mathcal{L}}$ is a collection of all distributions whose moment generating function is finite in an open neighbourhood of zero and that are not bounded from above (or from below, see Remark 6).

Theorem 4 *Under $\mathcal{P}_{asympt}(\epsilon, \delta)$ operating on $\tilde{\mathcal{L}}$,*

$$\liminf_{\delta \rightarrow 0} \frac{E_a T_2(\delta)}{\log(1/\delta)} = \infty. \quad (26)$$

The proof of Theorem 4 follows from the Lemmas 1 and 2.

Remark 6 Above, we assumed that $\tilde{\mathcal{L}}$ contains distributions that are not bounded above. The analysis is essentially unchanged, if we had instead assumed that it contains distributions

that are not bounded from below. One way to see this is to note that finding the population with the lower mean is equivalent to finding the one with the higher mean in this two population setting. This is true even for $d > 2$ populations with minor adjustments in the analysis. Then, if the random variables of each population are unbounded from below, one may consider negative of these random variables which are then unbounded from above.

5.2 More general setting

It is easy to extend this analysis in a variety of ways. For example, suppose our interest, given two populations, is to identify the one with the smallest p th quantile. Recall that for any real valued random variable Z with distribution function $F_Z(\cdot)$, the quantile function is the inverse of $F_Z(\cdot)$, and in particular, p th quantile is given by

$$F_Z^{-1}(p) = \inf_{x \in \mathbb{R}} \{F_Z(x) \geq p\}.$$

The analysis easily generalises to handle such cases.

We consider random elements taking values in a general state space \mathcal{X} . Let h denote a mapping from a probability distribution on \mathcal{X} to the real line. The p th quantile is one example of such a mapping (when $\mathcal{X} = \mathbb{R}$) from a probability distribution of a real-valued random variable to the real line. Let \mathcal{H} denote a collection of probability distributions on \mathcal{X} such that $h(G) < \infty$ for all $G \in \mathcal{H}$. Suppose that in comparing two distributions F and $G \in \mathcal{H}$, our aim is to identify $\min(h(F), h(G))$.

Now define a policy $\mathcal{P}(\epsilon, \delta)$ operating on distributions in \mathcal{H} with the property that for $F, G \in \mathcal{H}$ such that $|h(F) - h(G)| > \epsilon > 0$ the policy $\mathcal{P}(\epsilon, \delta)$, generates samples from the two distributions, adaptively deciding which distribution to sample from next and at some stage selects one of the two populations as the one with the lower h value. The policy guarantees that $P(FS) \leq \delta$. As before, $\mathcal{P}_{asympt}(\epsilon, \delta)$ guarantees this asymptotically.

In the discussion at the beginning of Section 5.1 where the two populations $(X_i : i \geq 1)$ and $(Y_i : i \geq 1)$ are compared under the two probability measures P_a and P_b , replace μ_F, μ_G and $\mu_{\tilde{G}}$ with $h(F), h(G)$ and $h(\tilde{G})$. Then, as in Lemma 1, it follows that under $\mathcal{P}_{asympt}(\epsilon, \delta)$,

$$\liminf_{\delta \rightarrow 0} \frac{E_a T_2(\delta)}{\log(1/\delta)} \geq \sup_{\tilde{G} \in \mathcal{H}, h(\tilde{G}) > h(F) + \epsilon, \tilde{G} \sim G} \frac{1}{3I(G, \tilde{G})}. \quad (27)$$

Remark 7 In the case where $\mathcal{X} = \mathbb{R}$, $h(F)$ denotes the p th quantile of F , it easy to see that RHS on (27) can be infinite. We show this when \mathcal{H} includes mixture of Gaussian distributions. Consider $p \leq 1/2$.

For $\mu > 0$, and $0 < \epsilon < p$, consider the pdfs

$$g(x) = \sqrt{\frac{1}{2\pi}} \left(p \exp(-x^2/2) + (1 - p) \exp(-(x - \mu)^2/2) \right),$$

and

$$g_\epsilon(x) = \sqrt{\frac{1}{2\pi}} \left((p - \epsilon) \exp(-x^2/2) + (1 - p - \epsilon) \exp(-(x - \mu)^2/2) \right).$$

Note that the p th quantile of g , $m_g < \mu/2$. The p th quantile of g_ϵ , $m_{g_\epsilon} \geq m(\epsilon)$ where $m(\epsilon)$ solves

$$\frac{\epsilon}{p + \epsilon} = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{m(\epsilon)} \exp(-(x - \mu)^2/2) dx = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{m(\epsilon) - \mu} \exp(-x^2/2) dx.$$

Using the bounds available for Gaussian tail probabilities, it can be seen that

$$\mu - m(\epsilon) \sim 2\sqrt{\log(p/\epsilon)}$$

as $\epsilon \rightarrow 0$. In particular it follows that for small ϵ , $m_{g_\epsilon} - m_g$ can be made arbitrarily large by increasing μ .

Now consider, the Kullback-Leibler distance

$$\int_{-\infty}^{\infty} \log \left(\frac{g(x)}{g_\epsilon(x)} \right) g(x) dx.$$

This easily seen to be bounded from above by $\log(p/(p - \epsilon))$ and can be made arbitrarily close to zero by choosing ϵ sufficiently close to zero. The case $p > 1/2$ is easily handled by considering $\mu < 0$ in definition of g and g_ϵ .

6 Positive results - the non-adaptive algorithms

In this section, we show that under conditions on moments of strictly increasing non-negative convex functions of the underlying populations, for any given $\epsilon, \delta > 0$, one can develop non-adaptive $\mathcal{P}(\epsilon, \delta)$ algorithms that require a deterministic and known $O(\log(1/\delta))$ computational effort for any instance of underlying populations. These algorithms rely on truncating random samples generated and carefully bounding the truncation error using explicitly available moment bounds.

Specifically, we suppose (as in the Introduction) that we can sample independently from d different random variables $(X(i) : i \leq d)$. Further, from each population i , we can generate independent identically distributed samples $(X(i, j) : j \geq 1)$. Recall that the the distribution of $(X(i) : i \leq d)$ is unknown and our aim is to find

$$i^* = \arg \min_{1 \leq j \leq d} EX(j).$$

First in Section 6.0.1, we assume that $X(i) \in [0, b]$ a.s for each $i \leq d$ for some $b > 0$ and review the $\mathcal{P}(\epsilon, \delta)$ algorithms that require $O(\log(1/\delta))$ computational effort in that simple

setting. Analysis is straightforward and is discussed (independently, it appears), e.g., in Even-Dar et. al. (2002, 2006), Glynn and Juneja (2004, 2011). In Section 6.1, we arrive at the maximum expected error that may result through either appropriately truncating or capping a random variable when an upper bound on the moment of strictly increasing, non-negative convex function of a random variable is explicitly known. In Section 6.1.1, we develop $\mathcal{P}(\epsilon, \delta)$ algorithms that require $O(\log(1/\delta))$ computational effort, when appropriate moment upper bounds are available for each population. Such bounds can often be found in simulation models by the use of Lyapunov function based techniques.

6.0.1 $\mathcal{P}(\epsilon, \delta)$ policy for bounded random variables

For $\epsilon, b > 0$, consider $\mathcal{X}_\epsilon(b) = \{(X(i) : i \leq d) : EX(i^*) < EX(j) - \epsilon \ \forall j \neq i^*, X(i) \in [0, b] \ \forall i\}$.

A reasonable algorithm on $\mathcal{X}_\epsilon(b)$ for a well chosen n (discussed later) is:

- Generate independent samples $(X(i, j) : i = 1, \dots, d \text{ and } j = 1, \dots, n)$.
- Let $\bar{X}(i) = \frac{1}{n} \sum_{j=1}^n X(i, j)$. Declare

$$\hat{i} = \arg \min_{1 \leq i \leq d} \bar{X}(i)$$

as the best design.

Recall that false selection occurs if $\hat{i} \neq i^*$, with probability

$$P(\bar{X}(i^*) > \min_{j \neq i^*} \bar{X}(j)).$$

This is bounded from above by

$$\sum_{j \neq i^*} P(\bar{X}(i^*) > \bar{X}(j))$$

Using Hoeffding's inequality, we have

$$P(\bar{X}(i^*) > \bar{X}(j)) \leq P(\bar{X}(i^*) - \bar{X}(j) - (EX(i^*) - EX(j)) > \epsilon) \leq \exp\left(-n \frac{\epsilon^2}{2b^2}\right).$$

Thus, $n = \frac{2b^2}{\epsilon^2} \log((d-1)/\delta)$ provides the desired $\mathcal{P}(\epsilon, \delta)$ policy for any set of populations in $\mathcal{X}_\epsilon(b)$.

6.1 Bounding the truncation error

Suppose that \mathcal{X} is a class of non-negative random variables and f is a strictly increasing non-negative convex function. Examples include $f(x) = x^\alpha$ for $x \geq 0$ and $\alpha > 1$, and $f(x) = \exp(\theta x)$ for $\theta > 0$. We discuss two formulations to bound errors resulting from 1) truncating a random variable, and 2) capping it.

Consider the optimization problem \mathbf{O}_1

$$\max_{X \in \mathcal{X}} EXI(X \geq u) \tag{28}$$

$$\text{such that} \quad Ef(X) \leq c, \tag{29}$$

for some positive u and c . Also consider \mathbf{O}_2 where the objective function is instead set to

$$\max_{X \in \mathcal{X}} E(X - u)I(X > u), \tag{30}$$

again under constraint (29).

Furthermore, since, by Jensen's inequality,

$$Ef(X) \geq f(EX) \geq f(0),$$

we assume that $c > f(0)$.

\mathbf{O}_1 denotes the worst case expected truncation error under constraint (29) when X is replaced by the truncated $XI(X < u)$. \mathbf{O}_2 denotes the smaller error when X is replaced by the capped $\min(X, u)$.

First observe that given any random variable X that satisfies (29), a two-valued random variable Y that takes value

- $E[X|X < u]$ ($E[X|X \leq u]$) with probability $P(X < u)$ ($P(X \leq u)$), and
- value $E[X|X \geq u]$ ($E[X|X > u]$) with probability $P(X \geq u)$ ($P(X > u)$),

has the same mean $EY = EX$ and same objective function values under \mathbf{O}_1 (\mathbf{O}_2), i.e.,

$$EYI(Y \geq u) = EXI(X \geq u), \quad (E(Y - u)I(Y > u) = E(X - u)I(X > u)).$$

Furthermore, $Ef(Y) \leq Ef(X)$, with equality only if $X = Y$ a.s. Thus, only random variables that take at most two values can solve our optimization problems \mathbf{O}_1 and \mathbf{O}_2 . It is also easy to see that at the optimal solution in both the cases, the constraint (29) has to be tight. Hence, we restrict our search to random variables taking values $0 \leq x_1 < x_2$ with probability $1 - p$ and $p \in [0, 1]$ where

$$(1 - p)f(x_1) + pf(x_2) = c,$$

so that $0 \leq x_1 < f^{-1}(c)$ and $x_2 \geq f^{-1}(c)$. Furthermore,

$$p = \frac{c - f(x_1)}{f(x_2) - f(x_1)}.$$

Propositions 3 and 4 below observe that unique (a.s.) solutions \tilde{X}_1 and \tilde{X}_2 , respectively, to the optimization problems **O1** and **O2** are either degenerate and equal $f^{-1}(c)$ with probability 1, or they take value zero with positive probability.

Proposition 3 *The unique optimal solution \tilde{X}_1 for **O1**,*

1. *equals $f^{-1}(c)$ with probability 1, for $u \leq f^{-1}(c)$.*
2. *For $u > f^{-1}(c)$, \tilde{X}_1 has a two-value distribution. It equals u with probability*

$$\frac{c - f(0)}{f(u) - f(0)},$$

and zero otherwise.

The following assumption considerably eases the analysis of **O2**.

Assumption 4 *Given any $u > 0$, there exists $x_u > u$, the unique solution to*

$$x - u = \frac{f(x) - f(0)}{f'(x)}.$$

The above assumption can be seen to hold, e.g., for $f(x) = x^\alpha$, $\alpha > 1$ and for $f(x) = \exp(\theta x)$, $\theta > 0$.

Proposition 4 *The unique optimal solution \tilde{X}_2 for **O2** under Assumption 4,*

1. *a.s. equals $f^{-1}(c)$ for $x_u \leq f^{-1}(c)$.*
2. *For $x_u > f^{-1}(c)$, \tilde{X}_2 has a two-valued distribution. It equals x_u with probability*

$$\frac{c - f(0)}{f(x_u) - f(0)},$$

and zero otherwise.

Remark 8 Suppose that $f(x) = x^\alpha$, $\alpha > 1$. Under **O1**, the solution corresponds to $\tilde{X}_1 = c^{1/\alpha}$ with probability 1 for $u \leq c^{1/\alpha}$ and the associated objective function value equals $c^{1/\alpha}$. Otherwise, \tilde{X}_1 takes two values 0 and u where the probability of the latter equals $cu^{-\alpha}$. The objective function value then equals $cu^{-(\alpha-1)}$.

Under \mathbf{O}_2 , for $u > 0$

$$x_u = u \left(\frac{\alpha}{\alpha - 1} \right).$$

Then, $\tilde{X}_2 = c^{1/\alpha}$ with probability 1 for $u \leq c^{1/\alpha} \left(\frac{\alpha-1}{\alpha} \right)$, and the associated objective function value equals $c^{1/\alpha} - u$. Otherwise, \tilde{X}_2 takes two values 0 and x_u where the probability of the latter equals

$$cu^{-\alpha} \left(\frac{\alpha - 1}{\alpha} \right)^\alpha.$$

The optimal objective function value equals

$$cu^{-(\alpha-1)} \left(\frac{(\alpha - 1)^{\alpha-1}}{\alpha^\alpha} \right). \quad (31)$$

Thus the worst case expected truncation error reduces by factor $\left(\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha} \right)$ if we use $\min(X, u)$ instead of $XI(X < u)$ to get random variables bounded from above by u .

Also note that under \mathbf{O}_1 if $f(x) = \exp(\theta x)$ for some $\theta > 0$, then $\tilde{X}_1 = \log c/\theta$ for $u \leq \log c/\theta$ with probability 1, and \tilde{X}_1 takes two values 0 and u where the probability of the latter equals

$$\frac{c - 1}{\exp(\theta u) - 1}.$$

The objective function value in this case equals $u \frac{c-1}{\exp(\theta u)-1}$.

Remark 9 In \mathbf{O}_1 and \mathbf{O}_2 , if we replace the constraint (29) with $Ef(X) \leq c$ with

$$Ef_i(X) \leq c_i$$

for $i = 1, \dots, d$, where each f_i is a convex function, and each c_i is a constant, then the previous analysis indicates that the solution search may without loss of generality be restricted to two-valued random variables. This remains true even if X is real valued but no longer restricted to be non-negative.

6.1.1 Analysis for unbounded random variables

We now return to the problem of finding $\mathcal{P}(\epsilon, \delta)$ algorithms that require a deterministic and known $O(\log(1/\delta))$ computational effort for any instance of underlying populations, when upper bounds on the moments of strictly increasing non-negative convex functions of the underlying random variables are explicitly known. We focus on using capping to bound the random variables, because this has smaller error compared to truncating.

Specifically, we make the following assumption:

Assumption 5 *There exists a non-increasing function R so that*

$$E(X(i) - R(x))I(X(i) > R(x)) \leq x,$$

for all $x \geq 0$.

Remark 10 To see sufficient conditions for this assumption to hold, suppose that we are given strictly convex, increasing and non-negative, twice differentiable functions $(f_i : i \leq d)$ and positive constants $(c_i : i \leq d)$ such that

$$Ef_i(X(i)) \leq c_i$$

for $i \leq d$.

Then,

$$E(X(i) - u)I(X(i) > u) \leq h_i(u)$$

where

$$h_i(u) \triangleq (x_{i,u} - u) \frac{c_i - f_i(0)}{f_i(x_{i,u}) - f_i(0)},$$

and $x_{i,u}$ uniquely satisfies

$$x_{i,u} - u = \frac{f_i(x_{i,u}) - f_i(0)}{f'_i(x_{i,u})}.$$

Differentiating both sides with respect to u , it follows that

$$\frac{f_i(x_{i,u}) - f_i(0)}{f'_i(x_{i,u})^2} f''_i(x_{i,u}) x'_{i,u} = 1.$$

In particular, $x'_{i,u} > 0$. Thus,

$$h_i(u) = \frac{c_i - f_i(0)}{f'_i(x_{i,u})}$$

is a strictly decreasing function of u . Let

$$r_i(x) \triangleq h_i^{-1}(x).$$

Then, $r_i(x)$ is a strictly decreasing function of x . In particular,

$$E(X(i) - r_i(x))I(X(i) > r_i(x)) \leq x,$$

for all $x \geq 0$. Observe that

$$R(x) = \max_{i \leq d} r_i(x)$$

satisfies Assumption 5.

The proposed $\mathcal{P}(\epsilon, \delta)$ algorithm that for $\beta \in (0, 1)$ requires at most

$$n_\beta = \frac{2R(\beta\epsilon)^2}{\epsilon^2(1-\beta)^2} \log((d-1)/\delta)$$

samples is:

- Generate independent samples $(X(i, j) : i = 1, \dots, d \text{ and } j = 1, \dots, n_\beta)$. Let $Y(i, j) = \min(X(i, j), R(\beta\epsilon))$ for all i, j .
- Let $\bar{Y}(i) = \frac{1}{n_\beta} \sum_{j=1}^{n_\beta} Y(i, j)$. Declare

$$\hat{i} = \arg \min_{1 \leq i \leq d} \bar{Y}(i)$$

as the best design.

The probability of false selection corresponds to

$$P(\bar{Y}(i^*) > \min_{j \neq i^*} \bar{Y}(j)). \quad (32)$$

Repeating the analysis in Section 6.0.1, keeping in mind that for $j \neq i^*$,

$$\begin{aligned} EY(i^*) - EY(j) &= E \min(X(i^*), R(\beta\epsilon)) - E \min(X(j), R(\beta\epsilon)) \\ &\leq EX(i^*) - (EX(j) - \beta\epsilon) \\ &\leq (1 - \beta)\epsilon, \end{aligned}$$

we conclude that (32) is bounded from above by δ .

Remark 11 The β that minimises n_β corresponds to that minimising $\frac{R(\beta\epsilon)^2}{(1-\beta)^2}$.

We solve this when $f_i(x) = x^\alpha$, $\alpha > 1$ for all i . Then, it can be seen from Remark 8 that

$$r_i(x) = \left(\frac{c_i}{x}\right)^{1/(\alpha-1)} \left(\frac{\alpha-1}{\alpha^{\alpha/(\alpha-1)}}\right)$$

and we can set

$$R(x) = \frac{1}{x^{1/(\alpha-1)}} \left(\frac{\alpha-1}{\alpha^{\alpha/(\alpha-1)}}\right) \max_{i \leq d} c_i^{1/(\alpha-1)}.$$

Then minimising n_β corresponds to maximising

$$\beta^{2/(\alpha-1)}(1-\beta)^2,$$

and is achieved at $\beta = 1/\alpha$. Thus the bound on the number of samples needed equals $n_{1/\alpha}$

or

$$\frac{2R(\beta\epsilon)^2}{\epsilon^2(1-\beta)^2} \log((d-1)/\delta) = 2 \left(\frac{\max_{i \leq d} c_i}{\epsilon^\alpha}\right)^{2/(\alpha-1)} \log((d-1)/\delta).$$

7 Sequential pure exploration algorithm

We now review some of the related literature that comes under the broad topic of stochastic multi-arm bandit methodology. We discuss the elegant sequential sampling strategy referred to as the successive elimination algorithm proposed by Even-Dar et. al. (2002, 2006). Although they also propose a slightly more effective median elimination algorithm in that paper, and there have been significant developments on the ordinal optimisation problem since then (see, e.g., Audibert and Bubeck 2010, Jamieson et. al. 2013), the sequential algorithm of Even-Dar et. al. (2006) is particularly simple and lends to an easier laconic discussion. They considered the setting where the underlying random variables were Bernoulli, while we allow generally distributed random variables when explicit bounds on the moments are available. As mentioned in the introduction, we use one of the proposed methods from Bubeck et. al. (2013) for this purpose that relies on careful truncation of underlying random variables. Their analysis focusses on the regret minimisation objective but is easily adapted to our pure exploration setting. We present a minor tweak - while they considered truncations of the form $XI(X < u)$ to bound rv X , we note the minor performance benefits from using instead the capped random variable $\min(X, u)$. These may also extend to the regret minimisation objective. In addition, we compute bounds on the expected number of samples generated under the pure exploration algorithm for general random variables.

7.1 Sequential algorithm

We refer to populations as arms in this section. Also, instead of finding an arm with minimum expected cost, in consonance with the literature, we focus of finding the one that maximises expected reward. Thus, we have d arms. Arm i when pulled gives a reward distributed as $X(i)$ and our aim is to find the best arm

$$i^* = \max_{i \leq d} EX(i).$$

Let the maximum above be achieved by a unique arm and let $\Delta_i = EX(i^*) - EX(i) > 0$ for all $i \neq i^*$.

Suppose that there exists a non-negative function α_m with the property that for $i \leq d$,

$$P(|\bar{X}_m(i) - EX(i)| > \alpha_m) \leq \frac{c\delta}{m^2d} \quad (33)$$

where $\bar{X}_m(i)$ denotes the average of m i.i.d. samples of $X(i)$, and

$$c = \left(\sum_{m=1}^{\infty} 1/m^2 \right)^{-1} = 6/\pi^2.$$

In the examples that we consider later α_m is seen to be decreasing for all $m \geq e$. In our discussions, we will ignore this minor issue and assume that α_m is a decreasing, non-negative function of m .

Equation (33) ensures that

$$P(E_\delta) \geq 1 - \delta,$$

where

$$E_\delta = \{|\bar{X}_m(i) - EX(i)| \leq \alpha_m, \forall m, \forall i \leq d\},$$

and is the rationale for the successive elimination algorithm outlined below.

Successive elimination algorithm

1. Set $m = 1$ and $S = \{1, 2, \dots, d\}$.
2. Set for each arm i , $\bar{X}_1(i) = 0$;
3. **Repeat**
 - Sample every arm $i \in S$ once and let $\bar{X}_m(i)$ be the average reward of arm i by trials or pulls m ;
 - Let $\bar{X}_m(\max) = \max_{i \in S} \bar{X}_m(i)$;
 - For each arm $i \in S$ such that $\bar{X}_m(\max) - \bar{X}_m(i) \geq 2\alpha_m$ **do**
 - set $S = S - \{i\}$;
 - end
 - $m = m + 1$;

Until $|S| > 1$;

It is easy to see that on the set E_δ the best arm is never eliminated and that all other arms are eventually eliminated (see Even-Dar et. al. 2002, 2006). Also, It can be easily checked using Hoeffding's inequality that $\alpha_m = b\sqrt{\frac{2}{m} \log\left(\frac{dm^2}{c\delta}\right)}$ works for random variables $X(i) - EX(i) \in [-b, +b]$, $i \leq d$ in (33).

7.1.1 Explicit moment bounds on random variables

The key step above was simply to arrive at the set $E_{i,\delta}$

$$= \{|\bar{X}_m(i) - EX(i)| < \alpha_m \text{ for all } m\},$$

that has probability at least $1 - \delta/d$.

Bubeck et. al. (2013) achieve this when each $X(i)$ has an explicit bound on its moment. They focus on the more interesting case where the bound is on α moment for $\alpha \in (1, 2]$. In this version we also restrict our discussion to this case. The key to their analysis is Lemma 3 below that relies on using truncation and Bernstein inequality (shown with the proof of Lemma 3). In Lemma 3, we also state the result when the random variables are capped instead of truncated.

Lemma 3 *Let $\delta \in (0, 1)$, $\alpha \in (1, 2]$, $K > 0$,*

$$p(\alpha) = (1 + \alpha) + (\sqrt{2} + 1/3),$$

$$\hat{p}(\alpha) = \frac{(\alpha - 1)^{\alpha-1}}{\alpha^\alpha} (1 + \alpha) + (\sqrt{2} + 1/3),$$

and $(X_i : i \leq n)$ be iid samples of rv X . Suppose that $E|X|^\alpha \leq K$ and let

$$B_m = \left(\frac{Km}{\log(\delta^{-1})} \right)^{\alpha-1}.$$

Then, with probability at least $1 - \delta$,

$$\bar{X}_n \leq EX + p(\alpha)K^{1/\alpha} \left(\frac{\log(\delta^{-1})}{n} \right)^{\frac{\alpha-1}{\alpha}}. \quad (34)$$

where

$$\bar{X}_n = \frac{1}{n} \sum_{m=1}^n X_m I(|X_m| \leq B_m)$$

denotes the empirical mean of truncated samples.

And,

$$\tilde{X}_n \leq EX + \hat{p}(\alpha)K^{1/\alpha} \left(\frac{\log(\delta^{-1})}{n} \right)^{\frac{\alpha-1}{\alpha}}. \quad (35)$$

where

$$\tilde{X}_n = \frac{1}{n} \sum_{m=1}^n \min(|X_m|, B_m),$$

denotes the capped empirical mean.

The reverse of (34)

$$\bar{X}_n \geq EX - p(\alpha)K^{(1+\epsilon)^{-1}} \left(\frac{\log(\delta^{-1})}{n} \right)^{\frac{\alpha-1}{\alpha}}.$$

also holds with probability at least $1 - \delta$ following essentially identical proof. Similarly, the reverse of (35).

7.2 Expected number of samples generated

Lemma below is useful to our analysis.

Lemma 4 *Suppose that $a \geq e, b \geq 1$ and $t = t^* \geq 1$ solves*

$$a + b \log t = t.$$

Then,

$$t^* \leq a + b \log a + \frac{2b^2}{a} \log(a + b).$$

To compute $ET(i)$ the expected number of times arm $i \neq i^*$ is pulled, note that

$$ET(i) = \sum_{m=1}^{\infty} P(\bar{X}_m(\max) - \bar{X}_m(i) < 2\alpha_m) \leq \sum_{m=1}^{\infty} P(\bar{X}_m(i^*) - \bar{X}_m(i) < 2\alpha_m).$$

For $i \neq i^*$, let

$$\tau_i^* = \inf\{m : 4\alpha_m \leq \Delta_i\}$$

(recall that $\Delta_i = EX(i^*) - EX(i)$). Then,

$$\begin{aligned} ET(i) &\leq \tau_i^* + \sum_{m=\tau_i^*+1}^{\infty} P(\bar{X}_m(i^*) - EX(i^*) - (\bar{X}_m(i) - EX(i)) < -2\alpha_m), \\ &\leq \tau_i^* + \sum_{m=\tau_i^*+1}^{\infty} (P(\bar{X}_m(i^*) \leq EX(i^*) - \alpha_m) + P(\bar{X}_m(i) \geq EX(i) + \alpha_m)) \\ &\leq \tau_i^* + 2\delta/d. \end{aligned}$$

Thus, the total expected number of samples generated for all arms $i \neq i^*$ is bounded from above by

$$\sum_{i \neq i^*} \tau_i + 2\delta$$

and the total number of samples generated is bounded from above by twice this amount.

7.2.1 Bounded random variables

In particular, when $X(i) - EX(i) \in [-b, +b]$, $\alpha_m = b\sqrt{\frac{2}{m} \log\left(\frac{dm^2}{c\delta}\right)}$ works. Let m^* be the solution to

$$4b\sqrt{\frac{2}{m} \log\left(\frac{dm^2}{c\delta}\right)} = \Delta_i.$$

Then, $\tau_i^* \leq m^* + 1$.

Hence, using Lemma 4,

$$\tau_i^* \leq \frac{32b^2}{\Delta_i^2} \log \left(\frac{d}{c\delta} \right) + \frac{64b^2}{\Delta_i^2} \log \left(\frac{32b^2}{\Delta_i^2} \log \left(\frac{d}{c\delta} \right) \right) + 1$$

plus terms that become small as δ decreases to zero.

Recall that the total number of samples generated is bounded from above by

$$2 \sum_{i \neq i^*} \tau_i^* + 4\delta.$$

Hence the dominant terms in the upper bound for total number of expected samples for small δ are

$$64b^2 \log \left(\frac{d}{c\delta} \right) \sum_{i \neq i^*} \frac{1}{\Delta_i^2}.$$

7.2.2 Explicit bound on moments

When $E|X(i)|^\alpha \leq K$ for all $i \leq d$, it is easily seen that

$$\alpha_m = \hat{p}(\alpha) K^{1/\alpha} \left(\frac{\log \left(\frac{2m^2 d}{c\delta} \right)}{m} \right)^{(\alpha-1)/\alpha}$$

satisfies (33).

Again, the number of times all the arms $i \neq i^*$ are pulled is bounded from above by

$$\sum_{i \neq i^*} \tau_i^* + 2\delta$$

where now τ_i^* is bounded from above by $m^* + 1$ and m^* solves the equation

$$4\hat{p}(\alpha) K^{1/\alpha} \left(\frac{\log \left(\frac{2m^2 d}{c\delta} \right)}{m} \right)^{(\alpha-1)/\alpha} = \Delta_i.$$

From Lemma 4, it follows that

$$m^* \leq a + b \log a + \frac{2b^2}{a} \log(a + b)$$

where

$$a = \left(\frac{4\hat{p}(\alpha) K^{1/\alpha}}{\Delta_i} \right)^{\frac{\alpha}{\alpha-1}} \log \left(\frac{2d}{c\delta} \right)$$

and

$$b = 2 \left(\frac{4\hat{p}(\alpha) K^{1/\alpha}}{\Delta_i} \right)^{\frac{\alpha}{\alpha-1}}.$$

Hence, the dominant terms in the upper bound for total number of expected samples for small δ are

$$2 \left(4\hat{p}(\alpha)K^{1/\alpha}\right)^{\frac{\alpha}{\alpha-1}} \log \left(\frac{2d}{c\delta}\right) \sum_{i \neq i^*} \left(\frac{1}{\Delta_i}\right)^{\frac{\alpha}{\alpha-1}}.$$

8 Appendix: Proofs

Proof of Theorem 1: Consider $P(\hat{I}_m(0) \geq a) = P(-\inf_{\theta} \hat{\Lambda}(\theta) \geq a)$. This is bounded from below by

$$\sup_{\theta \in \mathfrak{R}} P(\hat{\Lambda}_m(\theta) \leq -a)$$

Now

$$P(\hat{\Lambda}_m(\theta) \leq -a) = P\left(\frac{1}{m} \sum_{i=1}^m \exp(\theta X_i) \leq e^{-a}\right).$$

Then, from Cramer's Theorem (see Dembo and Zeitouni 1998, Corollary 2.2.19),

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log P\left(\frac{1}{m} \sum_{i=1}^m \exp(\theta X_i) \leq e^{-a}\right) = -\mathcal{I}_{\theta}(e^{-a}),$$

so that

$$\liminf_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{I}_m(0) \geq a) \geq -\inf_{\theta \in \mathfrak{R}} \mathcal{I}_{\theta}(e^{-a}).$$

To prove the upper bound

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{I}_m(0) \geq a) \leq -\inf_{\theta \in \mathfrak{R}} \mathcal{I}_{\theta}(e^{-a}),$$

in light of Lemma 5 below, we need to show that for any $\theta_1 < 0 < \theta_2$,

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P\left(\inf_{\theta \in [\theta_1, \theta_2]} \hat{\Lambda}_m(\theta) \leq -a\right) \leq -\inf_{\theta \in [\theta_1, \theta_2]} \mathcal{I}_{\theta}(e^{-a}) \quad (36)$$

We show the above for $\theta_1 = 0$. The case where $\theta_1 < 0$ and $\theta_2 = 0$ follows analogously.

First observe that for $\tilde{\theta}_1 > 0$,

$$\left\{\inf_{\theta \in [0, \tilde{\theta}_1]} \hat{\Lambda}_m(\theta) \leq -a\right\} \subset \left\{\hat{\Lambda}'_m(0) \leq -a/\tilde{\theta}_1\right\}.$$

Recalling that $\hat{\Lambda}'_m(0) = \frac{1}{m} \sum_{i=1}^m X_i$, and Assumption 1, it follows that given any constant K , there exists $\tilde{\theta}_1$ such that

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P\left(\inf_{\theta \in [0, \tilde{\theta}_1]} \hat{\Lambda}_m(\theta) \leq -a\right) \leq -K.$$

It therefore suffices to show (36) for $0 < \theta_1 < \theta_2$. To this end, define $\theta_{i,n} = \theta_1 + \frac{i\delta}{n}$ for $i = 0, 1, \dots, n$, where $\delta = \theta_2 - \theta_1$. Let

$$A_{i,n} = [\theta_{i,n}, \theta_{i+1,n}]$$

for $i = 0, 1, \dots, n-1$. Then,

$$P(\inf_{\theta \in [\theta_1, \theta_2]} \hat{\Lambda}_m(\theta) \leq -a) \leq \sum_{i=0}^{n-1} P(\inf_{\theta \in A_{i,n}} \hat{\Lambda}_m(\theta) \leq -a).$$

Observe by Jensen's inequality that for $\theta \in A_{i,n}$,

$$\hat{\Lambda}_m(\theta) \geq \frac{\theta}{\theta_{i,n}} \hat{\Lambda}_m(\theta_{i,n}).$$

Thus, if $\hat{\Lambda}_m(\theta_{i,n}) < 0$, since $\frac{\theta_{i+1,n}}{\theta_{i,n}} \leq \frac{\theta_1 + \delta/n}{\theta_1}$,

$$\inf_{\theta \in A_{i,n}} \hat{\Lambda}_m(\theta) \geq \frac{\theta_1 + \delta/n}{\theta_1} \hat{\Lambda}_m(\theta_{i,n}).$$

Hence,

$$P(\inf_{\theta \in A_{i,n}} \hat{\Lambda}_m(\theta) \leq -a) \leq P\left(\hat{\Lambda}_m(\theta_{i,n}) \leq -\frac{a\theta_1}{\theta_1 + \delta/n}\right).$$

Using Chernoff's bound, the RHS in turn is bounded from above by

$$\exp\left(-m \inf_{\theta \in [\theta_1, \theta_2]} \mathcal{I}_\theta\left(e^{-\frac{a\theta_1}{\theta_1 + \delta/n}}\right)\right).$$

We then have

$$P(\inf_{\theta \in [\theta_1, \theta_2]} \hat{\Lambda}_m(\theta) \leq -a) \leq n \exp\left(-m \inf_{\theta \in [\theta_1, \theta_2]} \mathcal{I}_\theta\left(e^{-\frac{a\theta_1}{\theta_1 + \delta/n}}\right)\right),$$

so that

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\inf_{\theta \in [\theta_1, \theta_2]} \hat{\Lambda}_m(\theta) \leq -a) \leq -\inf_{\theta \in [\theta_1, \theta_2]} \mathcal{I}_\theta\left(e^{-\frac{a\theta_1}{\theta_1 + \delta/n}}\right).$$

Observing that the derivative $\frac{\partial}{\partial \theta} \mathcal{I}_\theta(x)$ is continuous both in θ and x , it follows that

$$\inf_{\theta \in [\theta_1, \theta_2]} \mathcal{I}_\theta\left(e^{-\frac{a\theta_1}{\theta_1 + \delta/n}}\right) \rightarrow \inf_{\theta \in [\theta_1, \theta_2]} \mathcal{I}_\theta(e^{-a})$$

as $n \rightarrow \infty$. This concludes the proof of (4). \square .

Lemma 5 *There exist $\theta_1 < 0 < \theta_2$ such that*

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\inf_{\theta \in (\theta_2, \infty)} \hat{\Lambda}_m(\theta) \leq -a) \leq -k \quad (37)$$

and

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\inf_{\theta \in (-\infty, \theta_1)} \hat{\Lambda}_m(\theta) \leq -a) \leq -k, \quad (38)$$

where $k > \inf_{\theta \in \mathbb{R}} \mathcal{I}_\theta(e^{-a})$.

Proof of Lemma 5: To see (37), observe that

$$\left\{ \inf_{\theta \in (\theta_2, \infty)} \hat{\Lambda}_m(\theta) \leq -a \right\} \subset \left(\{ \hat{\Lambda}_m(\theta_2) \leq -a \} \cup \{ \hat{\Lambda}'_m(\theta_2) \leq 0 \} \right)$$

Recall that

$$\hat{\Lambda}'_m(\theta_i) = \frac{\frac{1}{m} \sum_{i=1}^m X_i \exp(\theta X_i)}{\frac{1}{m} \sum_{i=1}^m \exp(\theta X_i)}. \quad (39)$$

Now due to strict convexity of $\Lambda(\theta)$, $\Lambda(\theta)$ and $\Lambda'(\theta)$ are positive and increase with θ for all θ sufficiently large (they may become infinite). Thus, one can find $\theta_2 > 0$ so that

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{\Lambda}_m(\theta_2) \leq -a) \leq -k$$

and

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{\Lambda}'_m(\theta_2) \leq 0) \leq -k \quad (40)$$

To see (40), observe that $P(\hat{\Lambda}'_m(\theta_2) \leq 0)$ equals

$$P\left(\sum_{i=1}^m X_i \exp(\theta_2 X_i) \leq 0\right)$$

and the function $x \exp(\theta_2 x)$ is bounded from below for $\theta_2 > 0$. Equation (38) follows similarly. \square

Proof of Proposition 1:

Consider $0 < \theta_1 < \underline{\theta}_a < \bar{\theta}_a < \theta_2$ such that $\theta_1, \theta_2 \in \mathcal{D}_\Lambda^\circ$. Recall that

$$P(\hat{I}_m(0) \leq a) = P(\inf_{\theta \in \mathbb{R}} \hat{\Lambda}_m(\theta) \geq -a)$$

and the RHS is greater than or equal to

$$P\left(\inf_{\theta \in [\theta_1, \theta_2]} \hat{\Lambda}_m(\theta) \geq -a\right) - P\left(\inf_{\theta \in (-\infty, \theta_1]} \hat{\Lambda}_m(\theta) < -a\right) - P\left(\inf_{\theta \in [\theta_2, \infty)} \hat{\Lambda}_m(\theta) < -a\right).$$

From Corollary 1, it is clear that the last two probability terms in RHS above are bounded above by an exponentially decaying term in m as $m \rightarrow \infty$. It thus suffices to show that

$$\liminf_{m \rightarrow \infty} \frac{1}{m} \log P\left(\inf_{\theta \in [\theta_1, \theta_2]} \hat{\Lambda}_m(\theta) \geq -a\right) \geq 0 \quad (41)$$

(Since any probability is bounded from above by 1, the reverse direction is trivially true.)

To see (41), note that by Jensen's

$$\hat{\Lambda}_m(\theta) \geq \frac{\theta}{\theta_1} \hat{\Lambda}_m(\theta_1)$$

for $\theta > \theta_1$. hence, it follows that

$$P(\inf_{\theta \in [\theta_1, \theta_2]} \hat{\Lambda}_m(\theta) \geq -a) \geq P(\hat{\Lambda}_m(\theta_1) \geq -a\theta_1/\theta_2).$$

The RHS, in turn is bounded from below by

$$P(X_1 > \log m/\theta_1 - a/\theta_2) \geq m^{-\lambda/\theta_1} \exp(\lambda a/\theta_2).$$

Thus, (41) follows.

To see (7), observe that for $\theta > 0$,

$$P(e^{\theta X} \geq x) \geq x^{-\lambda/\theta}$$

for all x sufficiently large, so that $E \exp(\alpha e^{\theta X}) = +\infty$ for $\alpha, \theta > 0$.

Recall that $\mathcal{I}_\theta(e^{\Lambda(\theta)}) = 0$ for all $\theta \in \Theta$. For $e^x \geq e^{\Lambda(\theta)}$,

$$\mathcal{I}_\theta(e^x) = \sup_{\alpha \in \mathbb{R}} (\alpha e^x - \log E \exp(\alpha e^{\theta X})) = \sup_{\alpha \geq 0} (\alpha e^x - \log E \exp(\alpha e^{\theta X})) = 0.$$

(α above can be restricted to be non-negative for $e^x \geq e^{\Lambda(\theta)}$; see Dembo and Zeitouni 1998).

In particular, for $-a > -I(0) = \inf_{\theta \in \mathbb{R}} \Lambda(\theta)$,

$$\sup_{\theta \in [\underline{\theta}_a, \bar{\theta}_a]} \mathcal{I}_\theta(e^{-a}) = 0.$$

□

Proof of Theorem 2:

Some notation is useful for this proof.

Let \tilde{P} be another probability measure under which $(X_i : i \geq 1)$ remain iid, and their distribution is given as

$$\tilde{P}(X_i \in A) = \frac{E[\exp(\alpha^* e^{\theta^* X}) I(A)]}{E[\exp(\alpha^* e^{\theta^* X})]}.$$

Let $E_{\tilde{P}}$ denote the associated expectation operator. Then, (10) implies that $E_{\tilde{P}} X_i e^{\theta^* X_i} = 0$ under Assumption 3.

Note that we need to show that the lower bound

$$\liminf_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{I}_m(0) \leq a) \geq -\mathcal{I}_{\theta^*}(e^{-a}) \quad (42)$$

holds.

Recall that

$$P(\hat{I}_m(0) \leq a) = P(\inf_{\theta \in \mathbb{R}} \hat{\Lambda}_m(\theta) \geq -a)$$

As in proof of Proposition 1, consider $0 < \theta_1 < \underline{\theta}_a < \bar{\theta}_a < \theta_2$ such that $\theta_1, \theta_2 \in \mathcal{D}_\Lambda^a$. It then suffices to show that

$$\liminf_{m \rightarrow \infty} \frac{1}{m} \log P(\inf_{\theta \in [\theta_1, \theta_2]} \hat{\Lambda}_m(\theta) \geq -a) \geq -\mathcal{I}_{\theta^*}(e^{-a}). \quad (43)$$

Let $\hat{\theta} = \max(\theta_2 - \theta^*, \theta^* - \theta_1)$. Observe that for any $\epsilon > 0$,

$$P(\inf_{\theta \in [\theta_1, \theta_2]} \hat{\Lambda}_m(\theta) \geq -a - \epsilon) \geq P(\hat{\Lambda}_m(\theta^*) - |\hat{\Lambda}'_m(\theta^*)| \hat{\theta} > -a - \epsilon), \quad (44)$$

$$\geq P(\hat{\Lambda}_m(\theta^*) \geq -a, |\hat{\Lambda}'_m(\theta^*)| < \epsilon/\hat{\theta}), \quad (45)$$

$$\geq P\left(\hat{\Lambda}_m(\theta^*) \geq -a, \left|\frac{1}{m} \sum_{i=1}^m X_i e^{\theta^* X_i}\right| < \epsilon e^{-a}/\hat{\theta}\right), \quad (46)$$

Note that for any $\delta > 0$, $P(\hat{\Lambda}_m(\theta^*) \geq -a)$ equals

$$P\left(\hat{\Lambda}_m(\theta^*) \geq -a, \left|\frac{1}{m} \sum_{i=1}^m X_i e^{\theta^* X_i}\right| \leq \delta\right) + P\left(\hat{\Lambda}_m(\theta^*) \geq -a, \left|\frac{1}{m} \sum_{i=1}^m X_i e^{\theta^* X_i}\right| > \delta\right).$$

Easy to see that (as in proof of Cramer Theorem's upper bound),

$$P\left(\hat{\Lambda}_m(\theta^*) \geq -a, \left|\frac{1}{m} \sum_{i=1}^m X_i e^{\theta^* X_i}\right| > \delta\right) \leq \exp(-m\mathcal{I}_{\theta^*}(e^{-a})) \tilde{P}\left(\left|\frac{1}{m} \sum_{i=1}^m X_i e^{\theta^* X_i}\right| > \delta\right).$$

Since $E_{\tilde{P}} X_i e^{\theta^* X_i} = 0$, it follows that there exists a $\kappa > 0$ such that

$$\tilde{P}\left(\left|\frac{1}{m} \sum_{i=1}^m X_i e^{\theta^* X_i}\right| > \delta\right) \leq 2e^{-\kappa m}.$$

Now, from Cramer's Theorem,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{\Lambda}_m(\theta^*) \geq -a) = -\mathcal{I}_{\theta^*}(e^{-a}).$$

Thus also for any $\delta > 0$,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log P\left(\hat{\Lambda}_m(\theta^*) \geq -a, \left|\frac{1}{m} \sum_{i=1}^m X_i e^{\theta^* X_i}\right| \leq \delta\right) = -\mathcal{I}_{\theta^*}(e^{-a}).$$

Hence,

$$\liminf_{m \rightarrow \infty} \frac{1}{m} \log P(\inf_{\theta \in [\theta_1, \theta_2]} \hat{\Lambda}_m(\theta) \geq -a - \epsilon) \geq -\mathcal{I}_{\theta^*}(e^{-a}).$$

Letting $\epsilon \searrow 0$, we get

$$\liminf_{m \rightarrow \infty} \frac{1}{m} \log P(\inf_{\theta \in [\theta_1, \theta_2]} \hat{\Lambda}_m(\theta) \geq -a) \geq -\mathcal{I}_{\theta^*}(e^{-a}).$$

so that the result follows.

□

Proof of Theorem 3:

Note that there exists n_0 such that for $\epsilon > 0$ and small,

$$P\left(\sum_{i=1}^n X_i > 0\right) \geq \exp(-nI(0)(1 + \epsilon))$$

To get the lower bound, observe that for $a > I(0)$ and $\epsilon \in (0, I(0))$,

$$P\left(\sum_{i=1}^N X_i > 0\right) \geq E(\exp(-NI(0)(1 + \epsilon)I(N \geq n_0))) \quad (47)$$

$$\geq E(\exp(-NI(0)(1 + \epsilon)) - P(N \leq n_0)) \quad (48)$$

Below we show that

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(N \leq n_0) = -\infty. \quad (49)$$

Hence,

$$\liminf_{m \rightarrow \infty} \frac{1}{m} \log P\left(\sum_{i=1}^N X_i > 0\right) \geq \liminf_{m \rightarrow \infty} \frac{1}{m} \log E(\exp(-NI(0)))$$

This in turn, for $b > I(0), \epsilon \in (0, b - I(0))$, is

$$\geq \sup_{\theta} \liminf_{m \rightarrow \infty} \frac{1}{m} \log E \exp\left(\frac{c_2 m}{\hat{\Lambda}_m(\theta)} I(0)\right) \quad (50)$$

$$\geq \sup_{\theta} \liminf_{m \rightarrow \infty} \frac{1}{m} \log \exp\left(-\frac{c_2 m}{b - \epsilon} I(0)\right) \times P(\hat{\Lambda}_m(\theta) \in (-b - \epsilon, -b + \epsilon)) \quad (51)$$

$$\geq -\frac{c_2 I(0)}{b - \epsilon} - \inf_{\theta} \mathcal{I}_{\theta}(e^{-(b - \epsilon)}). \quad (52)$$

Since, the above is true for arbitrarily small $\epsilon > 0$ and for $b > I(0)$, the lower bound follows.

To see (15), for $c_1 = c_2 = 1$, observe that

$$\inf_{b \geq 0} \left(\frac{I(0)}{b} + \inf_{\theta} \mathcal{I}_{\theta}(e^{-a}) \right) \leq 1 + \inf_{\theta} \mathcal{I}_{\theta}(e^{-I(0)}).$$

Furthermore, for θ^* such that $I(0) = -\Lambda(\theta^*)$, we have $\mathcal{I}_{\theta^*}(e^{-I(0)}) = 0$, so that the RHS above is bounded from above by 1. Strict inequality in (15) can be inferred by differentiation.

To see (49), consider the probability $P(N \leq n_0)$. This is dominated by

$$P(\hat{I}_m(0) \geq m/n_0)$$

which in turn is bounded from above by

$$P(\hat{I}_m(0) \geq m_0/n_0)$$

for all $m \geq m_0$.

The following result is then useful.

Lemma 6 *Given any $K > 0$, there exists an $a > 0$ such that*

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{I}_m(0) \geq a) \leq -K.$$

Proof: Observe that

$$P(\hat{I}_m(0) \geq a) = P(\inf_{\theta \in \mathfrak{R}} \hat{\Lambda}(\theta) \leq -a)$$

Note that for any $\bar{\theta}$,

$$\{\inf_{\theta \geq \bar{\theta}} \hat{\Lambda}_m(\theta) \leq -a\} \subset \{\hat{\Lambda}_m(\bar{\theta}) \leq -a\} \cup \{\hat{\Lambda}'_m(\bar{\theta}) \leq 0\}$$

and for any $\underline{\theta}$

$$\{\inf_{\theta \leq \underline{\theta}} \hat{\Lambda}_m(\theta) \leq -a\} \subset \{\hat{\Lambda}_m(\underline{\theta}) \leq -a\} \cup \{\hat{\Lambda}'_m(\underline{\theta}) \geq 0\}$$

Thus,

$$P(\inf_{\theta \in \mathfrak{R}} \hat{\Lambda}(\theta) \leq -a) \leq 3P(\inf_{\theta \in [\underline{\theta}, \bar{\theta}]} \hat{\Lambda}(\theta) \leq -a) + P(\hat{\Lambda}'_m(\underline{\theta}) \geq 0) + P(\hat{\Lambda}'_m(\bar{\theta}) \leq 0).$$

Selecting $\underline{\theta}$ and $\bar{\theta}$ such that

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{\Lambda}'_m(\underline{\theta}) \geq 0) < -K$$

and

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\hat{\Lambda}'_m(\bar{\theta}) \leq 0) < -K$$

we now show that there exists a so that

$$\limsup_{m \rightarrow \infty} \frac{1}{m} \log P(\inf_{\theta \in [\underline{\theta}, \bar{\theta}]} \hat{\Lambda}(\theta) \leq -a) < -K, \quad (53)$$

which completes the proof. To see (53), observe that LHS equals

$$\inf_{\theta \in [\underline{\theta}, \bar{\theta}]} \mathcal{I}_\theta(e^{-a})$$

Furthermore, since

$$\mathcal{I}_\theta(e^{-a}) = \sup_{\alpha \in \mathfrak{R}} (\alpha e^{-a} - \log E \exp(\alpha e^{\theta X})) ,$$

by setting $\alpha = -e^a$, we get

$$\inf_{\theta \in [\underline{\theta}, \bar{\theta}]} \mathcal{I}_\theta(e^{-a}) \geq -1 - \sup_{\theta \in [\underline{\theta}, \bar{\theta}]} \log E[\exp(-e^{a+\theta X})]$$

The RHS is bounded from below by

$$-1 - \log E[\exp(-e^{a+\inf_{\theta \in [\underline{\theta}, \bar{\theta}]} \theta X})]$$

Since,

$$a + (\inf_{\theta \in [\underline{\theta}, \bar{\theta}]} \theta)X \rightarrow \infty$$

as $a \rightarrow \infty$, it follows by bounded convergence theorem that

$$\log E[\exp(-e^{a+\inf_{\theta \in [\underline{\theta}, \bar{\theta}]} \theta X})] \rightarrow -\infty$$

as $a \rightarrow \infty$. In particular, there exists an a sufficiently large so that (53) holds. \square

Proof of Proposition 2: Note that

$$P(FS) \geq P(\bar{X}_m \geq 0 \text{ and } \exp(-m\hat{I}_m(0)) \leq \delta)$$

for $m = c_1 \log(1/\delta)$. Thus, for (21) to hold, it suffices that

$$\liminf_{\delta \rightarrow \infty} \delta^{-1} P(\bar{X}_m \geq 0 \text{ and } \exp(-m\hat{I}_m(0)) \leq \delta) = \infty.$$

Equivalently,

$$\liminf_{\delta \rightarrow \infty} \delta^{-1} P(\hat{\Lambda}'_m(0) \geq 0 \text{ and } \inf_{\theta \in \mathfrak{R}} \hat{\Lambda}_m(\theta) \leq -1/c_1) = \infty.$$

Hence, for (21) to hold, it suffices to find $\theta < 0$ such that

$$\liminf_{\delta \rightarrow \infty} \delta^{-1} P(\hat{\Lambda}_m(\theta) \leq -1/c_1) = \infty.$$

Since, by Cramer's theorem, for $\Lambda(\theta) > -1/c_1$, for $\epsilon > 0$ and m sufficiently large,

$$P(\hat{\Lambda}_m(\theta) \leq -1/c_1) \geq \exp(-m(1+\epsilon)\mathcal{I}_\theta(e^{-1/c_1}))$$

(21) follows if

$$\liminf_{\delta \rightarrow \infty} \delta^{-1} \exp(-m(1+\epsilon)\mathcal{I}_\theta(e^{-1/c_1})) = \infty.$$

for some $\epsilon > 0$. This is implied by (20). \square

Proof of Lemma 1: Define

$$t^*(\delta) = \frac{1}{3I(G, \tilde{G})} \log(1/\delta).$$

Proof relies on a contradiction. Suppose that there exists a $\zeta \in (0, 1/2)$ such that

$$\liminf_{\delta \rightarrow 0} \frac{E_a T_2(\delta)}{t^*(\delta)} \leq 1 - \zeta. \tag{54}$$

Then we show that this implies that

$$\limsup_{\delta \rightarrow 0} P_b(J(\delta) = 2)\delta^{-1} = \infty,$$

providing the desired contradiction. so that (23) follows.

Note that (54) implies that, there exists $\eta \in (0, \zeta)$, and a sequence $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, such that

$$\sup_n \frac{E_a T_2(\delta_n)}{t^*(\delta_n)} \leq 1 - \eta.$$

Let $L(\delta)$ denote the likelihood ratio of P_b with respect to P_a of the generated samples until time $T(\delta) = T_1(\delta) + T_2(\delta)$.

Then,

$$P_b(J(\delta) = 2) = E_a[L(\delta)I(J(\delta) = 2)].$$

Since each X_i has the same distribution under P_a as well as P_b ,

$$L(\delta) = \prod_{i=1}^{T_2(\delta)} \frac{d\tilde{G}(Y_i)}{dG(Y_i)} = \exp \left(- \sum_{i=1}^{T_2(\delta)} \log \frac{dG(Y_i)}{d\tilde{G}(Y_i)} \right) \quad (55)$$

Let $S_1(\delta) = \{T_2(\delta) \leq 2t^*(\delta)\}$. Note by Markov's inequality that

$$P(S_1(\delta_n)^c) \leq (1 - \eta)/2.$$

By SLLN

$$\frac{1}{n} \sum_{i=1}^n \log \frac{dG(Y_i)}{d\tilde{G}(Y_i)} \rightarrow I(G, \tilde{G})$$

as $n \rightarrow \infty$, under P_a . In particular then,

$$\frac{1}{n} \max_{j \leq n} \sum_{i=1}^j \log \frac{dG(Y_i)}{d\tilde{G}(Y_i)} \rightarrow I(G, \tilde{G}),$$

so that

$$P \left(\max_{j \leq 2t^*(\delta_n)} \sum_{i=1}^j \log \frac{dG(Y_i)}{d\tilde{G}(Y_i)} \leq 2I(G, \tilde{G})(1 + \kappa)t^*(\delta_n) \right) \rightarrow 1$$

for any $\kappa > 0$ as $n \rightarrow \infty$.

Let $S_2(\delta) = \{\max_{j \leq 2t^*(\delta)} \sum_{i=1}^j \log \frac{dG(Y_i)}{d\tilde{G}(Y_i)} \leq I(G, \tilde{G})(1 + \kappa)2t^*(\delta)\}$.

Let $S_3(\delta) = \{J(\delta) = 2\}$ and let

$$\mathcal{E}(\delta) = S_1(\delta) \cap S_2(\delta) \cap S_3(\delta).$$

It can then be seen that $P_a(\mathcal{E}(\delta_n))$ for sufficiently large n is arbitrarily close to $(1 - \eta)/2$.

Now,

$$P_b(J(\delta_n) = 2) \geq P_b(\mathcal{E}(\delta_n)) = E_a[L(\delta_n)I(\mathcal{E}(\delta_n))].$$

Hence,

$$P_b(J(\delta_n) = 2) \geq P_a(\mathcal{E}(\delta_n)) \exp(-2I(G, \tilde{G})(1 + \kappa)t^*(\delta_n)) = P_a(\mathcal{E}(\delta_n))\delta_n^{2(1+\kappa)/3},$$

which for $\kappa < 1/2$ implies that $P_b(J(\delta_n) = 2)\delta_n^{-1} \rightarrow \infty$ as $n \rightarrow \infty$. This is the desired contradiction that implies (23). \square

Proof of Lemma 2: Consider a large b whose value will be fixed later. Furthermore, take $\gamma \in (0, 1)$. Construct G_k as follows: Set

$$G_k(x) = (1 - \gamma)G(x)$$

for all $x \leq b$, and,

$$\bar{G}_k(x) = \beta \bar{G}(x)$$

for $x > b$, where, $\bar{G}_k(x) = 1 - G_k(x)$ and $\bar{G}(x) = 1 - G(x)$.

Note that

$$\beta = 1 + \gamma \frac{G(b)}{\bar{G}(b)}.$$

Then,

$$\int_{x \in \mathbb{R}} \log \left(\frac{dG(x)}{dG_k(x)} \right) dG(x) = -G(b) \log(1 - \gamma) - \bar{G}(b) \log \beta \quad (56)$$

By selecting $\gamma = 1 - \exp(-\alpha/2)$, we get

$$-G(b) \log(1 - \gamma) \leq \alpha/2.$$

Furthermore, then $\bar{G}(b) \log \beta$ equals

$$\bar{G}(b) \log \left(\frac{1 - \exp(-\alpha/2)G(b)}{\bar{G}(b)} \right) \leq \bar{G}(b) \log \left(1 + \frac{\alpha G(b)}{2\bar{G}(b)} \right) < \alpha/2.$$

(Using $e^{-x} \geq 1 - x$ and $\log(1 + x) \leq x$ above).

Also, for b such that $G(b^+) = G(b^-)$,

$$\mu_{G_k} = (1 - \gamma) \int_{-\infty}^b x dG(x) + \left(1 + \gamma \frac{G(b)}{\bar{G}(b)} \right) \int_b^{\infty} x dG(x) \geq \exp(-\alpha/2) \mu_G + (1 - \exp(-\alpha/2)) G(b) b$$

Since, RHS increases to infinity as $b \rightarrow \infty$, one can select b sufficiently large so that $\mu_{G_k} \geq k$.

\square

Proof of Proposition 3: The case $u \leq f^{-1}(c)$, follows by noting that

$$f(EX) \leq Ef(X) \leq c,$$

so that $EXI(X \geq u) \leq EX \leq f^{-1}(c)$.

Now consider $u > f^{-1}(c)$. Suppose that \tilde{X}_1 takes values $x_1^* \in [0, f^{-1}(c))$ and $x_2^* \geq u$. ($x_2^* < u$ is not possible as then the optimal objective function value is zero.)

The objective function (28) at optimal value then equals

$$x_2^* \left(\frac{c - f(x_1^*)}{f(x_2^*) - f(x_1^*)} \right).$$

Note that

$$\frac{c - f(x)}{f(x_2^*) - f(x)}$$

for $x < f^{-1}(c)$ and $f(x_2^*) > c$, is a strictly decreasing function of x . Thus, $x_1^* = 0$.

Now observing that

$$f(y) - f(0) < yf'(y)$$

from simple calculus, it follows that

$$y \left(\frac{c - f(0)}{f(y) - f(0)} \right)$$

is non-increasing in y for $y \geq u$ and the result follows. \square

Proof of Proposition 4: Consider a solution that puts mass at two points x_2^* and $x_1^* \in [0, x_2^*)$ with probabilities p and $1-p$, respectively. Clearly, then $x_1^* < f^{-1}(c)$ and $x_2^* \geq f^{-1}(c)$.

First suppose that an optimal solution has $x_1^* \geq u$ with positive probability. We show that this leads to a contradiction. Note that $x_1^* \geq u$ only if $u < f^{-1}(c)$. In that case, the objective function equals

$$\left((x_2^* - x_1^*) \frac{c - f(x_1^*)}{f(x_2^*) - f(x_1^*)} + (x_1^* - u) \right) \quad (57)$$

Consider the function

$$\left((x - x_1^*) \frac{c - f(x_1^*)}{f(x) - f(x_1^*)} \right).$$

This is clearly non-increasing in x . Thus x_2^* in (57) equals $f^{-1}(c)$ with probability 1 providing the desired contradiction.

Now suppose that $x_1^* < u$. Then, to achieve a positive value of the objective function, we need $x_2^* > u$. In particular, the optimal objective function equals

$$\left((x_2^* - u) \frac{c - f(x_1^*)}{f(x_2^*) - f(x_1^*)} \right). \quad (58)$$

Note that

$$\frac{c - f(x)}{f(x_2^*) - f(x)}$$

for $x < \min(x_2^*, f^{-1}(c))$ and $f(x_2^*) \geq c$, is a strictly decreasing function of x . Thus, $x_1^* = 0$ in (58).

Now consider the function

$$\left((x - u) \frac{c - f(0)}{f(x) - f(0)} \right)$$

for $x > u$. Due to Assumption 4, this is maximised at x_u . Thus, if $x_u > f^{-1}(c)$, then clearly $x_2^* = x_u$ in (58). Else, $x_2^* = f^{-1}(c)$ with probability 1. The proposition stands proved. \square

Proof of Lemma 4: First note that $t^* \leq (a + b)^2$ for if not, then

$$(a + b) \log t^* > (a + b)^2$$

so that $\log t^* > (a + b)$. Since, $(a + b) \geq \frac{t^*}{\log t^*}$, it follows that $\log^2 t^* > t^*$ providing the desired contradiction (since $\log^2 t < t$ for $t \geq 1$).

Now

$$t^* = a + b \log t^* = a + b \log(a + b \log t^*) \leq a + b \log(a + 2b \log(a + b)).$$

and the result easily follows. \square .

The Bernstein inequality below is useful for the proof of Lemma 3.

Lemma 7 (Bernstein Inequality) *Suppose that $(X_i : i \geq 1)$ are iid mean zero rv and there exists $M > 0$:*

$$|X_i| \leq M.$$

Then,

$$P\left(\frac{1}{n} \sum_{i=1}^n X_i > t\right) \leq \exp\left(-\frac{n^2 t^2 / 2}{\sum_{i=1}^n EX_i^2 + Mnt/3}\right).$$

Proof of Lemma 3

Proof for (34) below is taken from Bubeck et. al. 2013.

We would like to show that

$$EX - \frac{1}{n} \sum_{m=1}^n X_m I(|X_m| \leq B_m) \tag{59}$$

is less than or equal to

$$p(\alpha) K^{1/\alpha} \left(\frac{\log(\delta^{-1})}{n} \right)^{\frac{\alpha-1}{\alpha}}$$

with probability at least $1 - \delta$. Note that (59) equals

$$\frac{1}{n} \sum_{m=1}^n (EX - E(XI(|X| \leq B_m))) + \frac{1}{n} \sum_{m=1}^n (E(XI(|X| \leq B_m)) - X_m I(|X_m| \leq B_m))$$

This in turn is less than or equal to

$$\frac{1}{n} \sum_{m=1}^n \frac{K}{B_m^{\alpha-1}} + \sqrt{\frac{2B_n^{2-\alpha} K \log(\delta^{-1})}{n}} + \frac{B_n \log(\delta^{-1})}{3n}, \quad (60)$$

with probability $1 - \delta$ where we used Bernstein inequality to bound

$$\frac{1}{n} \sum_{m=1}^n (E(XI(|X| \leq B_m)) - X_m I(|X_m| \leq B_m))$$

by

$$\sqrt{\frac{2B_n^{2-\alpha} K \log(\delta^{-1})}{n}} + \frac{B_n \log(\delta^{-1})}{3n},$$

observing that $EX^2I(|X| \leq B) \leq KB^{2-\alpha}$ in the inequality.

Equation (34) follows by plugging in the values of B_m in (60) and conducting some minor simplifications.

To see (35), in the above analysis, replace $X_m I(|X_m| \leq B_m)$ and $XI(|X| \leq B_m)$ by $\min(X_m, B_m)$ and $\min(X, B_m)$, respectively. Then, using (31), observe that (60) is replaced by

$$\left(\frac{(\alpha - 1)^{\alpha-1}}{\alpha^\alpha} \right) \frac{1}{n} \sum_{m=1}^n \frac{K}{B_m^{\alpha-1}} + \sqrt{\frac{2B_n^{2-\alpha} K \log(\delta^{-1})}{n}} + \frac{B_n \log(\delta^{-1})}{3n}, \quad (61)$$

Again, the result follows after simplifications. \square

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